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Quantum Theory of Spin-3/2 Field in Einstein Spaces[1]

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A quantum theory of a free massless spin-3/2 field on Einstein spaces ($R_{ab} = \Lambda g_{ab}$) is formulated in an algebraic framework. Attention is confined to the structure of the quantum operator algebra. In particular, the issue of positivity of the anticommutator is investigated and found to depend on whether or not the space-time admits "zero-frequency" neutrino solutions. Using methods developed for the purpose, a class of space-time that does not admit neutrino "zero modes" is characterized. An appendix introduces a useful technique for obtaining an initial value formulation of spinor field equations.

1. INTRODUCTION

The study of quantum fields on a fixed background space-time is an important step towards understanding quantum processes in the presence of gravity. It is of interest, therefore, to know what features of the underlying space-time give rise to new quantum phenomena. One particular direction to search for these new effects is to analyze the role of the global structure of space-time. For example, one may ask if the topology of the space-time is reflected in quantum field theory. Some authors (Sorkin, 1979; Ashtekar and Sen, 1980) have recently addressed this issue in the context of the Maxwell field. That they indeed find a new feature is due to the following happy circumstances. In a multiply connected space having "wormholes" or "handle" topology, one can associate a charge with each "handle"; it is in the structure of Maxwell's equation that the charges so defined stay constant in time as long as the topology does not change. At the quantum level,

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these charges arise as operators which form the center of the quantum operator algebra, i.e., the charges commute with every element of the algebra. This feature can be interpreted as a statement of charge superselection.

Are there other fields that show similar interesting quantum behavior? While in principle any field theory in curved space-time can be examined, the cost of complications in a detailed analysis limits the number of candidates to only a few. It seems reasonable to consider free (linear) fields and to expect that the qualitative features of their quantum behavior survive in more complicated nonlinear or "interacting" models. One example is the spin-3/2 field. In fact, the spin-3/2 field turns out to be the simplest nontrivial "higher spin" field which is amenable to treatment in curved space-time.

The purpose of this paper is to formulate and examine a quantum theory of a massless spin-3/2 field in curved space-time. (The field equation is the massless Rarita–Schwinger equation.) To avoid inconsistencies following from algebraic conditions on the field due to the curvature of space-time, one considers only space-times which are "Einstein spaces," i.e., solutions of the vacuum Einstein equation with cosmological constant. The quantum theory is formulated in an algebraic framework. In this approach, the "classical" field is regarded as a c-number spinor field on the space-time. The algebra of q-number operators is then constructed by isolating a preferred structure at the classical level. This structure is an inner product on the space of data and it determines the anticommutation relations. The main result which exposes the role of the space-time is that the anticommutation relation is positive definite if and only if the space-time does *not* admit solutions of the neutrino equation which are in a certain sense "zero-frequency" modes or "static." More precisely, the neutrino "zero modes" in a general space-time are elements of the kernel of a linear elliptic operator on a Cauchy slice Σ obtained by "$3+1$" decomposition of the neutrino equation relative to Σ and setting the time derivative to zero. It appears that the topology of Σ does not enter these considerations in any direct way. In the analysis of these issues, an initial value formulation of spinor field equations proves to be convenient. To this end, a notion of "$3+1$" decomposition of spinor fields and spinor calculus on the submanifold Σ has to be introduced. These techniques are of interest in their own right and the spin-3/2 theory that we consider here serves to illustrate their use.

In Section 2 we discuss the classical aspects of the spin-3/2 equation. In particular, the inner product on the space of data is defined and a criterion for its positivity is obtained. In Section 3 the algebra of quantum operators is constructed and in Section 4 the role of space-time in dictating its structure is discussed. We conclude with a brief discussion of the main results in Section 5. Appendix A introduces a technique for "3+1" decomposition of spinor equations and Appendix B sketches a proof that the propagation of the spin-3/2 field is causal.

2. THE SPIN-3/2 FIELD

2.1. Field Equation. Consider a globally hyperbolic, orientable space-time (M, g_{ab}).[2] Such space-times always admit a spinor structure (Geroch, 1970) thus enabling us to consider spinor fields on M. We shall work with two-component or Weyl spinors (rather than Dirac four-spinors) and our notation will be that of Pirani (1964).

A zero-rest-mass field of spin s is usually described by a totally symmetric spinor $\phi_{AB\cdots C}$ with $2s$ indices, $s>0$, satisfying the (conformally invariant) free field equations

$$\nabla^{AA'}\phi_{AB\cdots C}=0 \tag{1}$$

where $\nabla_{AA'}$ is the spinor form of the covariant derivative on (M, g_{ab}). This is the generalization of the flat space equations (Penrose 1965a). Thus, for example, the two-component neutrino equation is $\nabla^{A'A}\lambda_A=0$ and the source free Maxwell equation is $\nabla^{AA'}\phi_{AB}=0$, where λ_A is the neutrino field and ϕ_{AB} the Maxwell field, related to the skew tensor field $F_{ab}\equiv\phi_{AB}\varepsilon_{A'B'}+\bar{\phi}_{A'B'}\varepsilon_{AB}$. A spin-3/2 field according to this prescription is $\phi_{(ABC)}$ satisfying

$$\nabla^{AA'}\phi_{ABC}=0 \tag{2}$$

However, in nonflat space-times one encounters the following difficulty. Operating with $\nabla_{DD'}$ and contracting D', A' and D, B we obtain the algebraic condition known as a Buchdahl condition

$$\Psi_{ABCD}\phi^{ABC}=0 \tag{3}$$

where Ψ_{ABCD} is the spinor form of the Weyl tensor.[3] This is a strong restriction on the space-time and the field and in general will have very few solutions. Clearly one must look for a different spinor field with three indices satisfying some field equation in curved space-times. This is suggested by an alternative formalism for spin s fields in flat space which has

[2] Our conventions are the following: g_{ab} has signature $(+---)$ and the curvature tensors are defined by $[\nabla_a\nabla_b-\nabla_b\nabla_a]v_c=R_{abc}{}^{d}v_d$; $R_{ab}=R_{amb}{}^{m}$; and $R=R_{ab}g^{ab}$

[3] For the spinor form of curvature tensors see Pirani (1964).

been known for quite some time (Fierz and Pauli, 1939; Fierz, 1940; Garding, 1945). Here, a "spin"-s field is a spinor $\psi_{M'\cdots Q'A\cdots K}$ with $2s$ indices,[4] symmetric in both sets of indices and satisfying

$$\nabla^{AA'}\psi_{M'\cdots Q'A\cdots K}=0 \tag{4}$$

This equation suffers from the disadvantage that (in flat space) it is not irreducible under the inhomogeneous Lorentz group, having solutions which are a mixture of spin s, $s-1,\ldots,0$, or $1/2$. In *flat* space, the field $\psi_{M'\cdots Q'A\cdots K}$ satisfying (4) is in fact a potential for the field $\phi_{(M\cdots QA\cdots K)}$ satisfying (1), the two fields being related by $\phi_{(M\cdots QA\cdots K)}=\nabla_M{}^{M'}\cdots\nabla_Q{}^{Q'}\psi_{M'\cdots Q'A\cdots K}$. The lower spin parts of $\psi_{M'\cdots Q'A\cdots K}$ correspond to gauge freedom which does not affect the pure spin s field $\phi_{(M\cdots QA\cdots K)}$ (for details see Penrose, 1965a).

In particular, we consider a spin-3/2 field in curved space-time as a field $\psi_{A'(BC)}$ satisfying

$$\nabla^{BB'}\psi_{A'BC}=0 \tag{5}$$

This equation is in fact the massless Rarita–Schwinger equation[5] (Rarita and Schwinger 1941) extended to curved space-time. Are there any Buchdahl conditions that follow from equation (5)? It is easy to check that there are none if the space-time satisfies the vacuum Einstein equation with or without cosmological constant $R_{ab}=\Lambda g_{ab}$. For instance, applying $\nabla_{B'}{}^C$ to equation (5),

$$0=\nabla_{B'}{}^C\nabla^{BB'}\psi_{A'BC}=\Phi_{A'B'BC}\psi^{B'BC}$$

which is identically satisfied when $\Phi_{A'B'BC}$ (which is the spinor form of $R_{ab}-\frac{1}{4}Rg_{ab}$) vanishes.

Taking equation (5) as the field equation therefore forces one to a restricted class of space-times. In return for this loss of generality, however, one obtains a "sensible" quantum theory (as we shall presently show). One

[4] The numbers of unprimed and primed indices are unrelated. Only the total number of indices must be $2s$.

[5] The massive Rarita–Schwinger field is, in our notation, a pair $(\psi_{A'(BC)},\phi_{(A'B')C})$ satisfying the equations

$$\nabla_{A'}{}^A\psi_{B'AC}=(m/\sqrt{2})\phi_{A'B'C},\qquad \nabla_A{}^{A'}\phi_{A'B'C}=(m/\sqrt{2})\psi_{B'AC}$$

Setting $m=0$ one obtains a "two-helicity" massless spin-3/2 field. Here we are considering (for convenience) only a "one-helicity" spin-3/2 theory which corresponds to identifying $\phi_{A'B'C}$ with $\bar{\psi}_{CA'B'}$

could have considered an alternative, less stringent field equation

$$\nabla_{(B'}{}^{B}\psi_{A')CB}=0 \tag{5'}$$

which does not lead to any Buchdahl condition. The field described by (5′) can therefore be considered on arbitrary space-times. However, the main issue that argues against (5′) is that a "sensible" quantum theory cannot be formulated. As we shall indicate in the next section, the essential reason for this difficulty is that one would be forced to adopt a Hilbert space of states with an indefinite metric or, equivalently, the anticommutator of the field operators would be indefinite.

For completeness, we note that as a consequence of (5), $\psi_{A'BC}$ satisfies the wave equation

$$\Box\psi_{A'BC}-2\Psi_{BCMN}\psi_{A'}{}^{MN}+\tfrac{1}{6}R\psi_{A'BC}=0$$

To summarize, we consider spinor fields of the type $\psi_{A'(BC)}$ satisfying the equation $\nabla^{BB'}\psi_{A'BC}=0$ as our basic field of the spin-3/2 theory in space-times which satisfy $R_{ab}=\Lambda g_{ab}$.

2.2. "3+1" Decomposition of the Field Equations. The field $\psi_{A'(BC)}$ has six complex components and there are eight complex equations $\nabla^{BB'}\psi_{A'BC}=0$. In order to count the true dynamical degrees of freedom of the field, it is convenient to obtain an initial value formulation relative to a given Cauchy surface. In Appendix A, the technique for doing this is discussed. The basic idea is this: Fix a Cauchy surface Σ with an everywhere timelike future-directed unit normal vector field $t^{AA'}$. This vector field, regarded as a Hermitian spinor, provides us with a distinguished Hermitian inner product on the space of (say) unprimed spinors; equivalently, $t^{AA'}$ provides us with an isomorphism between primed and unprimed spinors at each point of Σ. In brief, the field $t^{AA'}$ on Σ enables us to convert primed indices into unprimed indices; the resulting unprimed spinor fields on Σ are in fact $SU(2)$ spinors. Since our field equation is first order, the data on Σ are just the field $\psi_{A'BC}$ restricted to Σ. In terms of $SU(2)$ spinors, the data are given by (see Appendix A)

$$\psi_{ABC}:=\sqrt{2}\,t_{A}{}^{A'}\psi_{A'BC} \tag{6}$$

ψ_{ABC} can be decomposed into irreducible pieces as

$$\psi_{ABC}=\psi_{A(BC)}=\psi_{(ABC)}-\tfrac{2}{3}\varepsilon_{A(B}\eta_{C)} \tag{7}$$

where

$$\eta_C := \varepsilon^{BA}\psi_{ABC} = \sqrt{2}\, t^{AA'}\psi_{A'AC} \tag{8}$$

In order to decompose the field equation into an equation on the hypersurface Σ, one needs to write the derivative operator in terms of a spatial derivative operator which refers only to the intrinsic geometry of Σ and a suitable time derivative. Such a decomposition involves the way Σ is embedded in the space-time and how one chooses to evolve the surface in time. As a result one has expressions involving both the extrinsic curvature π_{ab} of Σ and the lapse function N. Using techniques of Appendix A, one obtains the following equations in "3+1" form:

$$\frac{1}{\sqrt{2}} t\cdot\nabla\psi_{(ABC)} - \psi^{D(AB}\frac{D_{C)D}N}{N} + D_{M(A}\psi_{BC)}{}^{M} + \frac{\pi}{2\sqrt{2}}\psi_{(ABC)} + \frac{1}{\sqrt{2}}\pi_{M(AC}{}^{D}\psi_{B)D}{}^{M} - \frac{1}{\sqrt{2}}\pi_{M(AB}{}^{D}\psi_{|D|C)}{}^{M} = 0 \tag{9.1}$$

$$\frac{1}{\sqrt{2}} t\cdot\nabla\eta_A + D_{CM}\psi^{CM}{}_{A} - \frac{\pi_{CMDA}\psi^{CDM}}{\sqrt{2}} + \psi^{DB}{}_{A}\frac{D_{DB}N}{N} + \frac{\pi}{\sqrt{2}}\eta_A = 0 \tag{9.2}$$

$$D_{CM}\psi_B{}^{CM} + \frac{1}{\sqrt{2}}\pi_{CMDB}\psi^{DCM} = 0 \tag{9.3}$$

D_{AB} is the spatial derivative operator and $t\cdot\nabla = t^{MM'}\nabla_{MM'}$.[6] The first two equations are evolution equations for the data $(\psi_{(ABC)}, \eta_A)$ while the last equation is a constraint equation. A convenient form of equation (9.3) in terms of the pair $(\psi_{(ABC)}, \eta_A)$ is

$$D^{AB}\psi_{(CAB)} + \frac{\pi_{ABCD}}{\sqrt{2}}\psi^{(DAB)} + \frac{2}{3}\left[-D_{CM}\eta^{M} + \frac{\pi}{2\sqrt{2}}\eta_C\right] = 0 \tag{9.3$'$}$$

Since there are two constraint equations on the data, only four of the six complex functions can be freely specified as initial data. Of the six

[6] A convenient notion of time derivative for the spinor fields on Σ, given by P. Sommers (1980), is $\dot{\eta}_A := t\cdot\nabla\eta_A - \eta^B D_{AB}N/N$. With this definition, $\dot{\varepsilon}_{AB} = 0$ and for $v_a \equiv v_{(AB)}$ any spatial vector, $\dot{V}_{(AB)} = \dot{V}_a = h_a{}^b t\cdot\nabla V_b$. Since we shall not have occasion to use the evolution equations explicitly, we leave the time derivative in the primitive form $t\cdot\nabla$.

evolution equations, two are required to preserve the constraint, thus leaving four dynamical equations for four independent data. As shown in the next section, two of the four degrees of freedom are "gauge"; the remaining two are then the true dynamical degrees of freedom of the field.

2.3. Space of Pure Spin-3/2 Fields. Let V denote the space of solutions of the field equation (5). Linearity of the equation implies V is a vector space (over $\mathbb{C}$). It is easy to check that V has a proper subspace whose elements are of the form $\nabla_{A'(B}\lambda_{C)}$, where λ_C is a neutrino field, i.e., $\nabla^{CC'}\lambda_C=0$.[7] Thus, every neutrino solution gives us a solution of our field equation. This is a spin-1/2 contribution to the field and may be removed by imposing a gauge condition. However, there is no canonical way of fixing a particular gauge. This is analogous to the situation in electromagnetism where the freedom to add a gradient of a function to the vector potential cannot be removed by a covariant gauge condition (Strocchi, 1967). One might call the spin-1/2 piece the "longitudinal gravitino" in analogy with the "longitudinal photon."

In order to remove the longitudinal contribution we consider equivalence classes of fields $[\psi_{A'BC}]$, where two fields $\psi_{A'BC}$ and $\tilde{\psi}_{A'BC}$ are equivalent if and only if

$$\tilde{\psi}_{A'BC}-\psi_{A'BC}=\nabla_{A'(B}\lambda_{C)} \tag{10}$$

where λ_C is some neutrino field. These equivalence classes represent the pure spin-3/2 contribution of the original field $\psi_{A'(BC)}$.[7]

We now give a precise definition of the space of pure spin-3/2 field in terms of its data induced on a Cauchy slice Σ. Let τ denote the space of all pairs $(\psi_{(ABC)},\eta_A)$ of C^∞ spinor fields on Σ which satisfy the constraint equation (9.3′) and which are square integrable in the norm

$$\langle u,u\rangle=\int_\Sigma\left[\psi^{(ABC)\dagger}\psi_{ABC}+\tfrac{1}{3}\eta^{A\dagger}\eta_A\right]d\Sigma,\qquad u=(\psi_{(ABC)},\eta_A)$$

where $d\Sigma$ is the volume element defined by $d\Sigma_{BB'}=t_{BB'}d\Sigma$. $\psi^{(ABC)\dagger}$ stands for $(2)^{3/2}t^{AA'}t^{BB'}t^{CC'}\bar{\psi}_{(A'B'C')}$ and $\eta^{A\dagger}=\sqrt{2}\,t^{AA'}\bar{\eta}_{A'}$. (The factor 1/3 in the second term in the integrand is only for later convenience.) Denote by τ_0 the subspace of τ consisting of ("pure gauge") data pairs of the form $(t_{(A}{}^{A'}\nabla_{B|A'|}\lambda_{C)},t\cdot\nabla\lambda_C)$, where λ_C is a neutrino solution of compact support on Σ. The space $W=\tau/\bar{\tau}_0$, where $\bar{\tau}_0$ is the completion of τ_0 in the norm $\langle\ ,\ \rangle$, will be called the space of pure spin-3/2 data. Elements of $\bar{\tau}_0$ will be called pure gauge data.

[7]See, for example, de Wet (1940); Rarita and Schwinger (1941).

It is this space W which will be relevant for the quantum theory. However, before we can construct the quantum algebra, an additional structure is needed at the classical level. This structure is an inner product on W which we study in the following sections.

2.4. Inner Product on W. There is a preferred inner product on the space of solution, V, which induces an inner product on W. Fix a Cauchy surface Σ. Then for any two solutions $\psi_{A'BC}, \tilde{\psi}_{A'BC} \in V$, we define a product

$$\gamma(\psi, \tilde{\psi}) := (-\sqrt{2}) \int_\Sigma \bar{\psi}^{AA'B'} \tilde{\psi}_{A'A}{}^{B} \, d\Sigma_{BB'} \tag{11}$$

[The factor $(-\sqrt{2})$ is for later convenience.] In terms of the data on Σ corresponding to $\psi_{A'BC}$ and $\tilde{\psi}_{A'BC}$, viz., $u = (\psi_{(ABC)}, \eta_A)$ and $v = (\tilde{\psi}_{(ABC)}, \tilde{\eta}_A)$, (11) is given by

$$\gamma(u, v) = \int_\Sigma \left[\psi^{(ABC)\dagger} \tilde{\psi}_{(ABC)} - \tfrac{1}{3} \eta^{A\dagger} \tilde{\eta}_A \right] d\Sigma \tag{12}$$

By virtue of the field equation, the integrand on the right-hand side of (11) is a conserved current and therefore the definition of $\gamma(\,,)$ is independent of choice of Σ. Furthermore, although the inner product defined by (12) appears as a sesquilinear map γ: $\tau \times \tau \to \mathbb{C}$, $\gamma(\,,)$ is in fact a map from $W \times W$ to $\mathbb{C}$. To see this we show that $\gamma(v, u_0) = 0$, $\forall v \in \tau$, $u_0 \in \bar{\tau}_0$ [Then, under gauge transformation $u \to \tilde{u} = u + u_0$, $\gamma(v, \tilde{u}) = \gamma(v, u)$, i.e., $\gamma(\,,)$ is an inner product on gauge equivalence classes.] Consider $\hat{u}_0 = \nabla_{A'(B}\lambda_{C)}$ where λ_C is a neutrino field of compact support on Σ. Since for a neutrino field λ_C, $\nabla_{C'C}\lambda^C = 0$, $\nabla_{C'[C}\lambda_{A]} = \frac{1}{2}\varepsilon_{CA} \nabla_{C'M}\lambda^M = 0$. Hence

$$\nabla_{A'B}\lambda_A = \nabla_{A'(B}\lambda_{A)}$$

Next, using (11)

$$\gamma(\hat{v}, \hat{u}_0) = \sqrt{2} \int_\Sigma \bar{\psi}^{(A'B')A} \nabla_{A'(A}\lambda_{B)} \, d\Sigma_{B'}{}^{B}$$

$$= \sqrt{2} \int_\Sigma \bar{\psi}^{(A'B')A} \nabla_{A'B}\lambda_A \, d\Sigma_{B'}{}^{B}$$

$$= \sqrt{2} \int_\Sigma \nabla_{A'B} \left(\bar{\psi}^{(A'B')M} \lambda_M \right) d\Sigma_{B'}{}^{B}$$

for all solutions $\hat{v} = \psi_{A'(BC)}$ of the field equation (5). In the last step we have integrated by parts and used the field equation. Define a (complex) bivector $F^{[ab]} \equiv \bar{\psi}^{(A'B')M}\lambda_M \varepsilon^{AB}$. Then

$$\gamma(\hat{v}, \hat{u}_0) = \sqrt{2} \int_\Sigma \nabla_a F^{[ab]} d\Sigma_b = 0$$

In the last step we have used Stokes' theorem and that $F^{[ab]}$ is of compact support. Thus, if $u_0 \in \tau_0$, $v \in \tau$, respectively denote the data for $\hat{u}_0$ and $\hat{v}_0$, $\gamma(v, u_0) = 0$ $\forall v \in \tau$, $u_0 \in \tau_0$. Here γ is being used in the sense of (12), i.e., as the induced inner product on the data. Finally, since γ is continuous in the $\langle , \rangle$ norm, $\gamma(v, u_0) = 0$, $\forall v \in \tau$, $u_0 \in \bar{\tau}_0$.

Note that each term in the integrand in (12) is manifestly positive definite (since, for example, $\eta^{A\dagger}\eta_A \geqslant 0$). However, since the terms occur with opposite sign, it is not clear that γ is of a definite sign.

2.5. Positivity of $\gamma(,)$. Henceforth, we shall treat γ as a map γ: $W \times W \to \mathbb{C}$. As we shall see in the next section, this map plays a central role in the construction of the algebra of quantum operators. Furthermore, the physical interpretation of the resulting quantum theory hinges crucially on whether or not γ is positive. In this section we find a criterion that ensures positivity of γ.

While the detailed analysis of this problem is somewhat involved, the basic idea is quite simple. To obtain a sufficient condition for positivity of γ, the strategy is to use the gauge invariance of

$$\gamma(u, u) = \int_\Sigma \left[\psi^{(ABC)\dagger}\psi_{(ABC)} - \tfrac{1}{3}\eta^{A\dagger}\eta_A \right] d\Sigma \tag{13}$$

under addition of a neutrino contribution $(t_{(A}{}^{A'} \nabla_{B|A'|}\lambda_{C)}, t \cdot \nabla \lambda_C)$ to $u = (\psi_{(ABC)}, \eta_A)$. If we can find a neutrino datum λ_C such that

$$\left(t_{(A}{}^{A'} \nabla_{B|A'|}\lambda_{C)} + \psi_{(ABC)}, t \cdot \nabla \lambda_C + \eta_C \right) = \left(\tilde{\psi}_{(ABC)}, 0 \right) \in \tau \tag{14}$$

then $\gamma(u, u) = \int_\Sigma \tilde{\psi}^{(ABC)\dagger}\tilde{\psi}_{ABC} d\Sigma$, which is manifestly positive. Thus, the positivity of γ is ensured if, for a given η_C,

$$t \cdot \nabla \lambda_C + \eta_C = 0 \tag{15}$$

can always be solved for a neutrino datum λ_C. Since λ_C is a neutrino *datum*, $t \cdot \nabla \lambda_C$ actually stands for $-\sqrt{2}[D_{AC}\lambda^A + (\pi/2\sqrt{2})\lambda_C]$ (π is the trace of the extrinsic curvature of Σ) which is evident from the "3+1" form of the neutrino equation (A.33). Thus (15) can be expressed in the form

$$(L\lambda)_A := D_{AB}\lambda^B + \left(\pi/2\sqrt{2}\right)\lambda_A = \eta_A \tag{16}$$

We now proceed to obtain conditions that ensure the existence of solutions of (16).

Consider the space of smooth spinor fields η_A of compact support on Σ with a norm defined by $\langle \eta, \tilde{\eta} \rangle = \int_\Sigma \eta^{A\dagger} \tilde{\eta}_A \, d\Sigma$. Denote this space by H and let $\bar{H}$ be its Cauchy completion. Let $L := D_{AB} \varepsilon^{BC} + (\pi/2\sqrt{2}) \delta_A{}^C$ be a linear operator on H defined on the dense domain $D(L) = H$, and let L^* be the adjoint of L in the norm $\langle , \rangle$.

> *Lemma 1.* If $\operatorname{Ker} L^* = \{0\}$ then for every $\eta_A \in \bar{H}$ there exists a sequence $\{\lambda^n_A\}$ in H such that $\lim (L\lambda^n)_A = \eta_A$ and conversely.

Proof. Consider $\bar{H} = \overline{\operatorname{Im} L} \oplus (\operatorname{Im} L)^\perp$. Now, $\psi \in \operatorname{Ker} L^* \Leftrightarrow \forall \eta \in D(L)$ $\langle \psi, L\eta \rangle = 0 \Leftrightarrow \psi \in (\operatorname{Im} L)^\perp$. Hence $\operatorname{Ker} L^* = \{0\} \Leftrightarrow (\operatorname{Im} L)^\perp = \{0\} \Leftrightarrow \bar{H} = \overline{\operatorname{Im} L}$. In other words, $\operatorname{Ker} L^* = \{0\}$ iff every element of $\bar{H}$ can be obtained as the limit of a Cauchy sequence in $\operatorname{Im} L$.

The condition $\operatorname{Ker} L^* = \{0\}$ is also necessary for the positivity of γ. Suppose $\operatorname{Ker} L^* \neq \{0\}$; then we can always find data of the form $(0, \eta_A)$, $\eta_A \neq 0$ because the constraint equation (9.3′) reduces to $(L^*\eta)_A = 0$ when $\psi_{(ABC)}$ is zero. For the data $(0, \eta_A) = u$,

$$\gamma(u, u) = -\tfrac{1}{3} \int_\Sigma \eta^{A\dagger} \eta^A \, d\Sigma$$

which is manifestly negative.

Conversely, we show that if $\operatorname{Ker} L^* = \{0\}$ then $\gamma(u, u) = 0 \Rightarrow u \in \bar{\tau}_0$ i.e., u is pure gauge datum. Let $u = (\psi, \eta)$ and choose a sequence of data $u^n = (\tilde{\psi}^n, \tilde{\eta}^n)$ given by

$$\tilde{\psi}^n_{(ABC)} = \psi_{(ABC)} + t_{(A}{}^{A'} \nabla_{B|A'|} \lambda^n_{C)}$$

$$\tilde{\eta}^n_C = \eta_C + t \cdot \nabla \lambda^n_C$$

such that $\operatorname{Lim} \tilde{\eta}^{nA\dagger} \tilde{\eta}^n_A = 0$. (This is possible by the lemma.) Then since $\gamma(u, u) = \gamma(u^n, u^n) = 0$, $\operatorname{Lim} \tilde{\psi}^{n(ABC)\dagger} \tilde{\psi}^n{}_{(ABC)} = 0$ so $\operatorname{Lim} \tilde{\eta}^n{}_A = 0$ and $\operatorname{Lim} \tilde{\psi}^n{}_{(ABC)} = 0$. In other words $(\psi, \eta) = (\lim t_{(A}{}^{A'} \nabla_{B|A'|} \lambda^n_{C)}, \lim t \cdot \nabla \lambda^n_C)$, which is in fact an element of $\bar{\tau}_0$. We have shown the following.

> *Theorem.* A necessary and sufficient condition for $\gamma(,)$ to be positive definite on W [i.e., $\gamma(u, u) \geqslant 0$ equality holding iff $u = 0$, $\forall u \in W$] is $\operatorname{Ker} L^* = \{0\}$

What can we say about γ without assuming $\operatorname{Ker} L^* = \{0\}$? Quite generally any data $(\psi, \eta) \in \tau$ can be written as

$$(\psi, \eta) = (\psi, \eta_1) + (0, \eta_2)$$

where $\eta_2 \in \operatorname{Ker} L^*$ and $\eta_1 \in (\operatorname{Ker} L^*)^{\perp}$. Since (ψ, η) is a datum and $(0, \eta_2)$ is a datum (ψ, η_1) is also a datum, i.e., it satisfies the constraint equation. Further, as we saw earlier, $(\operatorname{Ker} L^*)^{\perp} = \overline{\operatorname{Im} L}$. Hence we conclude that (ψ, η_1) is equivalent (by a suitable choice of gauge) to data of the form $(\tilde{\psi}, 0)$. Hence the (gauge) equivalence class $[(\psi, \eta)]$ containing the datum $(\psi, \eta) \in \tau$ can be written as

$$[(\psi, \eta)] = [(\tilde{\psi}, 0)] + [(0, \eta_2)]$$

Denote by W_1 the (vector) space of $[(\tilde{\psi}, 0)]$ and by W_2 the (vector) space of $[(0, \eta_2)]$. Clearly for $u_1 \in W_1$, $u_2 \in W_2$, $\gamma(u_1, v_1)$ is positive definite, $\gamma(u_2, v_2)$ is negative definite, and $\gamma(u_1, v_2) = 0$. To summarize, if $\operatorname{Ker} L^* \neq \{0\}$ then (W, γ) can still be canonically decomposed (with respect to Σ) into two orthogonal subspaces: $W = W_1 \oplus W_2$ such that γ is positive (negative) definite on $W_1(W_2)$.

3. QUANTUM THEORY

We shall formulate the quantum theory in an algebraic framework in order to bring out certain features that are not manifest in other approaches, e.g., functional methods. Specifically, our main concern will be to construct and examine the algebra of quantum operators rather than to study the representation of the algebra on a Hilbert space of states. It turns out that even at the level of quantum algebra, a possible role of the space-time is exposed.

To obtain the algebra of operators we shall adopt an approach similar to that developed by Ashtekar and Magnon (1975) and others (Kay, 1977; Moreno, 1977; Magnon-Ashtekar, 1978) for scalar fields in curved space-time. As in the case of the scalar field an abstract quantum algebra can be constructred from information about the classical (c-number) field. By a classical Fermi field we shall mean a (c-number) spinor field (with appropriate index structure) satisfying a field equation. The Dirac, neutrino, and spin-3/2 fields are examples. In these cases there is available a preferred inner product γ on the (vector) space of solutions V, inherited from a local conserved current associated with the field equation. Then the Clifford algebra over (V, γ)[8] provides a natural way to obtain an (abstract)

[8]A Clifford algebra over (V, γ) is defined as follows. Let $\mathfrak{A}$ denote the tensor algebra generated by V: $\mathfrak{A} = \oplus_K^\infty V^{\otimes K}$. A typical element A of $\mathfrak{A}$ is a string with only a finite number of nonzero entries: $A = (\alpha, \alpha^a, \alpha^{ab}, \ldots, 0, 0 \ldots)$, where $\alpha^{a \cdots b}$ with K indices is the Kth-rank tensor over V. Sums and products of these strings are defined in the obvious way. Next consider elements of the form $C = (\gamma(t, \bar{t}), 0, t^a \bar{t}^b + t^b \bar{t}^a, 0, 0, \ldots)$. Let I denote the subalgebra generated by such elements, i.e., elements of I are sums of products of elements in $\mathfrak{A}$ containing at least one factor of the type C. I is in fact an ideal. The quotient algebra $\mathfrak{A}/I$ is called the Clifford algebra over (V, γ).

algebra of quantum operators. In the following section we shall display this construction (in somewhat more familiar form) for the spin-3/2 field.

We mention (but shall not explicitly show) that a Fock representation of the algebra can be obtained in a manner similar to the Bose cases. [For the construction of the Fock space for the Dirac field, for example, see Wald (1979).]

The Algebra of Spin-3/2 Field Operators. Let $\underline{F}_{A'BC}(x)$ denote the q number of "operator"-valued distribution satisfying the field equation

$$\nabla^{AA'}\underline{F}_{B'BA}=0 \tag{17}$$

The conjugate operator will be denoted by $\underline{F}^{*}_{AA'B'}$. Next fix a Cauchy hypersurface, Σ, and consider the space of smooth spinor test functions $\mathcal{H}=\{(\psi_{(ABC)},\eta_A)\}$ which are square integrable on Σ (in $\langle,\rangle$ norm) and define "smeared" field operators $\underline{F}(u)$ for all $u\in\mathcal{H}$ by

$$\underline{F}(u):=\gamma(u,\underline{F})=\int_{\Sigma}\left[\psi^{(ABC)\dagger}\underline{F}_{(ABC)}-\tfrac{1}{3}\eta^{A\dagger}\underline{f}_{A}\right]d\Sigma \tag{18}$$

where $(\underline{F}_{(ABC)},\underline{f}_A)$ is the "3+1" decomposition of $\underline{F}_{A'BC}$ relative to Σ and $u=(\psi,\eta)$. [Notice that the elements of $\mathcal{H}$ are not required to satisfy the constraint equation (9.3').] Denote by $\mathfrak{G}$ the free (*) algebra generated by $\underline{F}$ (and **1**). Next obtain an algebra $\mathfrak{A}\subset\mathfrak{G}$ of field operators as follows. Whenever u and v are square integrable *data*, impose on $\underline{F}(u)$ and $\underline{F}(v)$ the anticommutation relations

$$\begin{aligned}[\underline{F}(u),\underline{F}^{*}(v)]_{+}&=\gamma(u,v)\mathbb{1}\\ [\underline{F}(u),\underline{F}(v)]_{+}&=0\end{aligned} \tag{19}$$

The relations (19) are preserved under deformations of Σ in the following sense: If the data u and v evolve to $\hat{u}$ and $\hat{v}$ on a different Cauchy slice $\hat{\Sigma}$ then $\underline{F}(\hat{u})$ and $\underline{F}(\hat{v})$ satisfy (19) on $\hat{\Sigma}$ by virtue of the fact $\gamma(u,v)=\gamma(\hat{u},\hat{v})$.

By way of motivation for the anticommutation relations (19) we remark that for the Dirac field the anticommutation relations are precisely of the form given in (19), where the γ product of two Dirac fields $u=(\xi^A,\eta_{A'})$ and $v=(\lambda^A,\mu_{A'})$ is given by $\gamma(u,v)=\int_\Sigma(\xi^{A\dagger}\lambda_A+\bar{\eta}^{A\dagger}\bar{\mu}_A)\,d\sigma$.

The algebra $\mathfrak{A}$, as it stands, is too "large" in the sense that there is gauge freedom in the field operators.[9] The gauge transformations are

[9] We note that starting with a larger algebra $\mathfrak{A}$ is reminiscent of the situation in the Maxwell theory where one imposes commutation relations between all vector potentials and then selects out the physical subalgebra by the Gupta–Bleuler procedure (see, for example, Thirring, 1958).

expressed as automorphisms of the algebra $\mathfrak{A}$ generated by

$$\underline{F}_{A'BC} \overset{\alpha_\lambda}{\rightarrow} \underline{F}_{A'BC} + (\nabla\lambda)_{A'BC}\mathbb{1} \tag{20}$$

where $(\nabla\lambda)_{A'BC}$ is a pure gauge c-number field, i.e., the datum corresponding to this field is an element of τ_0 in this section. (The notation $\nabla\lambda$ is a reminder that data in τ_0 come from the field $\nabla_{A'(B}\lambda_{C)}$, λ_C a neutrino field of compact support on Σ.) α_λ is an automorphism because the anticommutation relations (19) are preserved under the transformations (20) (as can be easily checked). We write the action of the automorphism α_λ on the field operators $\underline{F}(u)$ as

$$\alpha_\lambda \circ \underline{F}(u) = \gamma(u, \underline{F} + \mathbb{1}\nabla\lambda) = \underline{F}(u) + \gamma(u, \nabla\lambda)\mathbb{1} \tag{21}$$

The gauge-invariant subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ is obtained as the fixed points of the gauge automorphisms:

$$\mathfrak{A}_0 = \{A : A \in \mathfrak{A}, \alpha_\lambda \circ A = A, \forall\lambda\} \tag{22}$$

We shall call $\mathfrak{A}_0$ the quantum spin-3/2 algebra. We shall shortly see that enlarging the gauge automorphisms to include $\nabla\lambda \in \bar{\tau}_0$ (rather than τ_0) does *not* affect $\mathfrak{A}_0$.

To gain some insight into the structure of the algebra we show that $\mathfrak{A}_0$ is generated by field operators of the type $\underline{F}(u)$, where u is *not* an arbitrary spinor test function on Σ but is in fact a *datum* for the classical field equation. It is clear that every datum u gives a field operator $\underline{F}(u)$ in $\mathfrak{A}_0$ since under a gauge transformation

$$\alpha_\lambda \circ \underline{F}(u) = \underline{F}(u) + \gamma(u, \nabla\lambda)\mathbb{1}$$

and $\gamma(u, \nabla\lambda) = 0$ when u is a datum. (See Section 2.4.) We now show that every generator of $\mathfrak{A}_0$ comes from some datum. First observe that with every $u = (\psi, \eta) \in \mathcal{H}$ we can associate a pair $\tilde{u} = (\psi, -\eta) \in \mathcal{H}$ so that

$$\langle u, v \rangle = \gamma(\tilde{u}, v) \qquad \forall v \in \mathcal{H} \tag{23}$$

where $\langle , \rangle$ is the positive definite norm defined in Section 2.3. Next, if $\underline{F}(\tilde{u}) \in \mathfrak{A}_0$, then $\alpha_\lambda \circ \underline{F}(\tilde{u}) = \underline{F}(\tilde{u})\ \forall\alpha_\lambda$ and so from (21) $\gamma(\tilde{u}, \nabla\lambda) = 0\ \forall\nabla\lambda \in \tau_0$. From (23) then $\langle u, \nabla\lambda \rangle = 0\ \forall\nabla\lambda \in \tau_0$. The datum is then of the form

$$\left(t_{(A}{}^{A'}\nabla_{B|A'|}\lambda_{C)}, t\cdot\nabla\lambda_C\right)$$
$$\equiv \left(D_{(AB}\lambda_{C)} - \frac{\pi}{\sqrt{2}}(ABC)D\lambda^D, -2\left[D_{CM}\lambda^M + \frac{\pi}{2\sqrt{2}}\lambda_C\right]\right)$$

where the first entry uses the definition of $D_{AB}\lambda_C$ and the second is obtained from the "3+1" decomposition of the neutrino equation (see Appendix A). Hence

$$\langle u, \nabla\lambda\rangle = \int_\Sigma \psi^{(ABC)\dagger}\left[D_{AB}\lambda_C - \frac{\pi}{\sqrt{2}}{}^{ABCD}\lambda^D\right]$$

$$+\frac{2}{3}\eta^{A\dagger}\left[-D_{AM}\lambda^M - \frac{\pi}{2\sqrt{2}}\dot{\lambda}_A\right]d\Sigma$$

Integrating by parts

$$\langle u, \nabla\lambda\rangle = \int_\Sigma (-1)\left[D^{AB}\psi_{(ABC)} + \frac{\pi}{\sqrt{2}}ABDC\psi^{(ABD)}\right.$$

$$\left.+\frac{2}{3}\left(DCM\eta^M - \frac{\pi}{2\sqrt{2}}\eta_C\right)\right]^\dagger \lambda^C\, d\Sigma \tag{24}$$

Demanding $\langle u, \nabla\lambda\rangle = 0$, $\forall\lambda$ then implies [from (24)]

$$D^{AB}\psi_{(ABC)} + \frac{\pi}{\sqrt{2}}{}^{ABDC}\psi^{(ABD)} + \frac{2}{3}\left(D_{CM}\eta^M - \frac{\pi}{2\sqrt{2}}\eta_C\right) = 0 \tag{25}$$

Comparing (25) with the constraint equation (9.3′) we find that if u satisfies (25) then $\tilde{u} = (\psi, -\eta)$ satisfies the constraint equation. That is, $\tilde{u}$ is a *datum*!

We have shown that $\gamma(\tilde{u}, \nabla\lambda) = \langle u, \nabla\lambda\rangle = 0\ \forall\nabla\lambda\in\tau_0 \Rightarrow \tilde{u}$ is a datum. Hence every generator $\underline{F}(\tilde{u})$ of $\mathfrak{A}_0$ comes from a datum $\tilde{u}$. Moreover, enlarging the gauge data to include $\nabla\lambda\in\bar{\tau}_0$ (instead of τ_0) does not affect $\mathfrak{A}_0$ (because $\bar{\tau}_0$ is the closure of τ_0 in $\langle\ ,\ \rangle$). Furthermore, from the discussion in Section 2 it follows that $\underline{F}(\tilde{u}) = \gamma(\tilde{u}, \underline{F}) = 0 \Rightarrow [\underline{F}(\tilde{u}), \underline{F}(v)] = 0\ \forall v \Leftrightarrow \gamma(\tilde{u}, v) = 0\ \forall v \Rightarrow \tilde{u}$ is "pure gauge," i.e., $\tilde{u}\in\bar{\tau}_0$. Thus to each element $u\in w = \tau/\bar{\tau}_0$ one can uniquely associate a field operator $\underline{F}(u)$. Then $\mathfrak{A}_0$ is indeed isomorphic to the Clifford algebra over (w, γ).[8]

The positivity of the anticommutator of field operators clearly depends on the positivity of $\gamma(,)$. In particular, if $\mathrm{Ker}\, L^* \neq \{0\}$ (see Section 2.5) then $W = W_1 \oplus W_2$ and $[\underline{F}(u_1), \underline{F}^*(u_1)]_+ \geq 0$, $[\underline{F}(u_1), \underline{F}^*(u_2)]_+ = 0$ and $[\underline{F}(u_2), \underline{F}^*(u_2)]_+ \leq 0$ for $u_1 \in W_1$ and $u_2 \in W_2$. The indefiniteness of the anticommutator in this instance raises questions about the physical interpretation of the quantum theory. We defer the discussion of this issue till Section 5.

To complete the analysis of the structure of the quantum algebra, we note that there are no elements in $\mathfrak{A}_0$ except zero which anticommutes with all generators of the algebra. This fact follows somewhat trivially from the way the field operators $\underline{F}(u)$ and the anticommutator are defined and has nothing to do with whether or not γ is positive definite. To see this, fix a datum u and the corresponding field operator $\underline{F}(u)$. Then $[\underline{F}(u), \underline{F}^*(v)]=0.\mathbb{1}$ for all v implies [from (19)] that $\gamma(u, v)=0$ for all v; hence $\gamma(u, \underline{F})=0$; i.e., $\underline{F}(u)=0$.

We conclude this section by indicating the essential problem one encounters with the alternative spin-3/2 theory described by equation (5′) and with higher spin fields. Equation (5′) can be written in the form

$$\nabla_{A'}{}^{A}\psi_{B'AC}=-\tfrac{1}{2}\varepsilon_{A'B'}\lambda_C, \qquad \nabla^{M'M}\psi_{M'MC}=\lambda_C$$
$$\nabla_{C'C}\lambda^C=\Phi_{C'B'CB}\psi^{B'CB}$$

which shows that the field $\psi_{A'BC}$ can be regarded as a mixture of "spin-3/2" and "spin-1/2" pieces. Note that in Einstein spaces, λ_C is in fact a neutrino field. Further there is a gauge freedom in the equations [as for equation (5)] which, in Einstein spaces, is again generated by a neutrino field. While no restriction on the space-time is required if we consider the field equation (5′), one has now a theory with several fields. It is not possible to isolate a pure spin-3/2 contribution from the field. One might still expect to formulate a physically sensible quantum theory. The quantum algebra is specified by the inner product $\gamma(\,,)$, which, in this case, is also given by (12). It turns out that the data $(\psi_{(ABC)}, \eta_A)$ for the field described by equation (5′) is not constrained: one can choose it arbitrarily. Thus the inner product $\gamma(u, u)=\int_\Sigma[\psi^{(ABC)\dagger}\psi_{(ABC)}-\frac{1}{3}\eta^{A\dagger}\eta_A]\,d\Sigma$ is indefinite. The anticommutator of the field operators is therefore indefinite and the quantum states of the theory would have to be elements of a Hilbert space with an indefinite metric. The physical interpretation of such a quantum theory is then not clear.

Similar problems plague the higher spin fields. The general fermion spin-s field equation consistent in arbitrary spacetimes (and derivable from a Lagrangian) is

$$\nabla_{(A'}{}^{A}\psi_{B'\cdots C')AB\cdots C}=0$$

where $\psi_{(B'\cdots C')(AB\cdots C)}$ is the spin-s field, having $2s$ indices, m of which are primed and $m+1$ unprimed (i.e., $2m+1=2s$) and symmetric in both sets of indices. Note that equation (5′) is a special case of this equation. The data for this field are again unconstrained and the inner product $\gamma(\,,)$ on the data is indefinite.

4. THE ROLE OF SPACE-TIME

In the previous section we saw that $\gamma(,)$ gives the anticommutator of the quantum operators. So the anticommutator is positive if $\gamma(,)$ is, and by the results of Section 2.5, this is ensured by the condition $\mathrm{Ker}\, L^* = \{0\}$. Here we discuss the restrictions on the choice of space-times implied by this requirement.

One faces some technical difficulties in working with the condition $\mathrm{Ker}\, L^* = \{0\}$. One would like to interpret the condition to mean

$$(\tilde{L}^*\lambda)_A := D_{AB}\lambda^B - \frac{\pi}{2\sqrt{2}}\lambda_A = 0, \quad \lambda_A \in C^\infty, \quad \int_\Sigma \lambda^{A+}\lambda_A \, d\Sigma < \infty \Rightarrow \lambda_A = 0 \tag{i}$$

That is, there are no square integrable solutions of the differential equation $(\tilde{L}^*\lambda)_A = 0$, where $\tilde{L}^*$ is the linear differential operator $D_{AB}\varepsilon^{BC} - (\pi/2\sqrt{2})\delta_A{}^C$. Now, since $\mathrm{Ker}\, L^*$ may well contain elements that are not smooth, it is not a priori obvious that the alternate statement (i) is equivalent to $\mathrm{Ker}\, L^* = \{0\}$. One must examine the properties of the operator $\tilde{L}^*$ to establish the equivalence of the two statements. In this instance, it is the ellipticity of $\tilde{L}^*$ that justifies (i). Briefly, the argument proceeds in two steps. First, one shows that $\mathrm{Ker}\, L^*$ consists of distribution solutions (defined below) of $(\tilde{L}^*\lambda)_A = 0$. This is done as follows. From the definition of distributions (see, for example, Reed and Simon, 1972) it is easy to check that for $\lambda_A \in \bar{H}$, the product $\langle \lambda, \eta \rangle = \int_\Sigma \lambda^{A+}\eta_A \, d\Sigma \ \forall \eta_A \in H$ defines a distribution $\underline{\lambda}: \underline{\lambda}(\eta) := \langle \lambda, \eta \rangle$. Now $\underline{\lambda}$ is said to be a distributional solution (or weak solution) of $(\tilde{L}^*\eta)_A = 0$ if the distribution $(\tilde{L}^*\underline{\lambda})$ defined by $(\tilde{L}^*\underline{\lambda})(\eta) := \underline{\lambda}(L\eta)$ satisfies $(\tilde{L}^*\underline{\lambda})(\eta) = 0 \ \forall \eta \in H$. Thus we see from $(\tilde{L}^*\underline{\lambda})(\eta) = \langle \lambda, L\eta \rangle = 0$ that every λ in $\mathrm{Ker}\, L^*$ gives a weak solution of $(L^*\eta)_A = 0$. In the second step, we establish that every weak solution is in fact smooth. One checks that $\tilde{L}^*$ is an elliptic differential operator with C^∞ coefficients (since π is smooth). [For the definition of elliptic differential operators see Atiyah and Singer (1963).] Then from the known result (Peetre, 1961) that every elliptic operator with C^∞ coefficients is hypoelliptic, it is immediate that every weak solution of $(\tilde{L}^*\eta)_A = 0$ is smooth. [A differential operator $\tilde{L}^*$ is said to be hypoelliptic if whenever $f \in C^\infty(\Omega)$ (Ω is an open set in Σ) and u is a weak solution of $\tilde{L}^*u = f$ on Ω, u must be C^∞ on Ω. For details see, for, example, Folland (1976), Reed and Simon (1972).] Thus one may identify $\mathrm{Ker}\, L^*$ with solutions of $(\tilde{L}^*\eta)_A = 0$.

In the following analysis we shall frequently integrate by parts and neglect contributions from the boundary terms. This imposes a restriction on the class of space-times that we can consider. The problem of boundary terms does not arise in the case of spatially closed space-times. Of the open space-times, we expect at least the asymptotically flat space-times to give no boundary contributions. This expectation is made plausible by observing that in Minkowski space, the leading term in the multipole expansion of solutions of $(\tilde{L}^*\eta)_A=0$ (that vanish at infinity) is of $O(1/r^2)$.[10] The relevant boundary terms that we encounter are then zero. Henceforth, we shall consider spatially closed or asymptotically flat space-times.

As before, consider a space-time (M, g_{ab}) whose Ricci curvature R_{ab} satisfies $R_{ab}=\Lambda g_{ab}$, $\Lambda=$const. Let Σ be a Cauchy slice with extrinsic curvature π_{ab} and metric [of signature $(---)$] h_{ab}. From the initial value formulation of general relativity (see, for example, Geroch, 1972) we know that the extrinsic curvature π_{ab} is constrained by the following equations:

$$-\mathfrak{R}-\pi^{ab}\pi_{ab}+\pi^2=2\Lambda \tag{26.1}$$

$$D_a(\pi^{ab}-\pi h^{ab})=0 \tag{26.2}$$

where $\mathfrak{R}$ is the scalar curvature of the three-manifold Σ. Then conditions (26.1) and (26.2) give us the following lemma.

Lemma 2. (i) For $\Lambda>0$, $(\tilde{L}^*\lambda)_A=D_{AB}\lambda^B-(\pi/2\sqrt{2})\lambda_A=0 \Leftrightarrow \lambda_A=0$, i.e., $\operatorname{Ker} L^*=\{0\}$

(ii) For $\Lambda=0$, $(\tilde{L}^*\lambda)_A=0 \Leftrightarrow D_{AB}\lambda_C+(\pi/\sqrt{2})ABCD\lambda^D=0$ for $\lambda_A \in \overline{H}$

Proof.[11]

$$\begin{aligned}
\langle(\tilde{L}^*\lambda)_A,(\tilde{L}^*\lambda)_A\rangle &= \left\langle D_{AB}\lambda^B-\frac{\pi}{2\sqrt{2}}\lambda_A,\, D_{AC}\lambda^C-\frac{\pi}{2\sqrt{2}}\lambda_A\right\rangle \\
&= \langle D_A{}^B\lambda_B,\, D_A{}^C\lambda_C\rangle+\tfrac{1}{8}\langle\pi\lambda_A,\pi\lambda_A\rangle \\
&\quad+\frac{1}{2\sqrt{2}}\left[\langle\pi\lambda_A,\, D_A{}^C\lambda_C\rangle+\langle D_A{}^B\lambda_B,\pi\lambda_A\rangle\right] \\
&= -\langle\lambda_B,\, D_B{}^A D_A{}^C\lambda_C\rangle+\tfrac{1}{8}\langle\lambda_A,\pi^2\lambda_A\rangle \\
&\quad+\frac{1}{2\sqrt{2}}\left[\langle\pi\lambda_A,\, D_A{}^C\lambda_C\rangle+\langle D_A{}^B\lambda_B,\pi\lambda_A\rangle\right] \\
&= \left\langle\lambda_B,\left(\frac{D^{MN}}{2}D_{MN}-\frac{\mathfrak{R}}{8}\right)\lambda_B\right\rangle \\
&\quad+\frac{1}{8}\langle\lambda_A,\pi^2\lambda_A\rangle-\frac{1}{2\sqrt{2}}\langle\lambda_B,(D_B{}^A\pi)\lambda_A\rangle
\end{aligned} \tag{27}$$

[10] In Minkowski space, if F_1 and F_2 are the components of a neutrino field in a basis, then the solutions of $D_{AB}\lambda^B=0$ (choosing a $\pi=0$ slice) are given by $F_1=(1/r)R_{+1/2}(r)S_{+1/2}(\theta,\phi)$, $F_2=(1/r)R_{-1/2}(r)S_{-1/2}(\theta,\phi)$ where $S_{\pm1/2}$ are of the form ${}_sY_{lm}(\theta,\phi)={}_sS_{lm}(\theta)e^{im\phi}$ (spin $s=\pm1/2$ weighted spherical harmonics) and $R_{-1/2}=A_l r^{l+1/2}+B_l r^{-(l+1/2)}$

$$R_{+1/2}=(1/r)R_{-1/2}, \qquad l\geq 1/2, \quad A_l \text{ and } B_l=\text{const}$$

[11] We use the notation $\langle\lambda_A,\eta_A\rangle:=\int_\Sigma \lambda^{A+}\eta_A\, d\Sigma$ etc.

To obtain the first term in the second step we have used the result (A.20), i.e., D_{AB} is skew symmetric in the norm $\langle\ ,\rangle$. The first term in (27) is obtained by simplifying $D_A{}^B D_{BC}$ using (A.29) and the last term is obtained by simple integration by parts. Next using (26.1) to write

$$-\mathfrak{R}+\pi^2=\pi^{ab}\pi_{ab}+2\Lambda$$

one obtains

$$\langle(\tilde{L}^*\lambda)_A,(\tilde{L}^*\lambda)_A\rangle=\frac{1}{2}\langle\lambda_B,D^{MN}D_{MN}\lambda_B\rangle+\frac{1}{8}\langle\lambda_B,\pi^{ab}\pi_{ab}\lambda_B\rangle$$
$$+\frac{\Lambda}{4}\langle\lambda_B,\lambda_B\rangle-\frac{1}{2\sqrt{2}}\langle\lambda_B,(D_B{}^A\pi)\lambda_A\rangle \qquad (28)$$

Using the second constraint equation (26.2) to replace $D_B{}^A\pi$ by $D_{CD}\pi^{CD}{}_B{}^A$ in the last term in (28) and integrating by parts,

$$\langle(\tilde{L}^*\lambda)_A,(\tilde{L}^*\lambda)_A\rangle=\frac{1}{2}\langle D_{MN}\lambda_B,D_{MN}\lambda_B\rangle+\frac{1}{8}\langle\pi_{ab}\lambda_A,\pi_{ab}\lambda_A\rangle$$
$$+\frac{\Lambda}{4}\langle\lambda_A,\lambda_A\rangle-\frac{1}{2\sqrt{2}}\left[\langle D_{MN}\lambda_B,\pi_{MNB}{}^A\lambda_A\rangle+\langle\pi_{MNB}{}^A\lambda_A,D_{MN}\lambda_B\rangle\right]$$

Since $\pi^{ABC}{}_D\pi_{ABCE}=\frac{1}{2}\varepsilon_{ED}\pi^{ab}\pi_{ab}$ it is easy to check

$$\langle\pi_{ab}\lambda_B,\pi_{ab}\lambda_B\rangle=2\langle\pi_{MNB}{}^A\lambda_A,\pi_{MNB}{}^A\lambda_A\rangle$$

Then

$$\langle(\tilde{L}^*\lambda)_A,(\tilde{L}^*\lambda)_A\rangle=\frac{1}{2}\langle D_{MN}\lambda_B,D_{MN}\lambda_B\rangle+\frac{1}{4}\langle\pi_{MNB}{}^A\lambda_A,\pi_{MNB}{}^A\lambda_A\rangle$$
$$-\frac{1}{2\sqrt{2}}\left[\langle D_{MN}\lambda_B,\pi_{MNB}{}^A\lambda_A\rangle\right.$$
$$\left.+\langle\pi_{MNB}{}^A\lambda_A,D_{MN}\lambda_B\rangle\right]+\frac{\Lambda}{4}\langle\lambda_A,\lambda_A\rangle$$
$$=\frac{1}{2}\left\langle\left(D_{MN}\lambda_B+\frac{\pi_{MNBA}}{\sqrt{2}}\lambda^A\right),\right.$$
$$\left.\times\left(D_{MN}\lambda_B+\frac{\pi_{MNBA}}{\sqrt{2}}\lambda^A\right)\right\rangle+\frac{\Lambda}{4}\langle\lambda_A,\lambda_A\rangle \qquad (29)$$

From (29) if $\Lambda > 0$, then since each term on the right is positive $(\tilde{L}^*\lambda)_A = 0 \Rightarrow \langle (\tilde{L}^*\lambda)_A, (\tilde{L}^*\lambda)_A \rangle = 0 \Rightarrow \langle \lambda_B, \lambda_B \rangle = 0 \Rightarrow \lambda_B = 0$. This completes the first part of the lemma. If $\Lambda = 0$, again (29) implies $(\tilde{L}^*\lambda)_A = 0 \Rightarrow D_{MN}\lambda_B + (\pi_{MNBA}/\sqrt{2})\lambda^A = 0$. This converse is trivial.

Lemma 3. The only vacuum globally hyperbolic closed or asymptotically flat space-time admitting a $\pi = \text{const}$ Cauchy slice for which $\operatorname{Ker} L^* \neq \{0\}$ is flat.

Proof. When $\pi = \text{const}$, from (28) (setting $\Lambda = 0$ and $D_{AB}\pi = 0$)

$$\langle (\tilde{L}^*\lambda)_A, (\tilde{L}^*\lambda)_A \rangle = \tfrac{1}{2}\langle D_{MN}\lambda_B, D_{MN}\lambda_B \rangle + \tfrac{1}{8}\langle \pi_{ab}\lambda_B, \pi_{ab}\lambda_B \rangle$$

$$(\tilde{L}^*\lambda)_A = 0 \Rightarrow D_{MN}\lambda_B = 0 \tag{30.1}$$

$$\pi_{ab}\lambda_B = 0 \tag{30.2}$$

From (30.1), $\lambda_B \in \operatorname{Ker} L^*$ must be a parallel spinor. Since Σ is three dimensional, existence of a nonzero parallel spinor implies Σ is flat.[12] Further, if λ_B is nonzero, (30.2) implies $\pi_{ab} = 0$. Hence $\operatorname{Ker} L^* \neq 0$ occurs only if there exists a $\pi = \text{const}$ Cauchy slice with a flat metric h_{ab} and $\pi_{ab} = 0$. Since the space-time is a solution of Einstein's equation $R_{ab} = 0$, (h_{ab} flat, $\pi_{ab} = 0$) constitutes initial data for flat space, i.e., the space-time is flat.

Lemmas 2 and 3 contain the central results concerning some space-times that admit a positive definite norm $\gamma(\,,)$. In the proofs of the lemmas the assumption that the boundary terms (in the various integrations by parts) vanish is crucial. Therefore our results apply to only those manifolds Σ which guarantee that the boundary terms vanish. Such manifolds will be said to have "negligible boundaries." Closed and, as we have indicated, asymptotically flat Cauchy hypersurfaces have negligible boundaries. One may well expect our results to fail when Σ has a boundary or is incomplete.[13]

[12] Any $SU(2)$ spinor $\lambda^A \in (V, \varepsilon_{AB})$ at $p \in \Sigma$ determined three orthogonal vectors (at p):

$$U^{(AB)} = \lambda^A\lambda^B + \lambda^{A+}\lambda^{B+}, \qquad V^{(AB)} = \lambda^A\lambda^B - \lambda^{A+}\lambda^{B+}, \qquad W^{(AB)} = \lambda^{A+}\lambda^B + \lambda^{B+}\lambda^A$$

Thus if λ^A is a parallel spinor field on Σ, then one has a parallel frame on Σ, implying Σ is flat. Alternatively, one shows that the integrability condition for $D_{AB}\lambda_C = 0$ requires that the Ricci curvature of Σ vanishes, which implies (since Σ is three dimensional) that the Riemann curvature vanishes, i.e., Σ is flat.

[13] In general, it is hard to characterize spaces with negligible boundary. One might begin with the observation that the boundary term is of the form $I = \int_\Sigma D_a V^a \, d\Sigma$ and its vanishing depends on the properties of V^a and on Σ. To find conditions such that $I = 0$, the following result due to Gaffney (1954) might be useful: On an orientable, complete, Riemannian manifold with C^2 Riemann tensor, if V^a is a C^1 vector field with the property that both $\|V\|$ and $D_a V^a$ are integrable (i.e., $\int_\Sigma \|V\| \, d\Sigma < \infty$ etc.), then $\int_\Sigma D_a V^a \, d\Sigma = 0$. There seems to be no obvious way in which this result can be applied to our situation.

From part (i) of Lemma 2, all space-times which are solutions of $R_{ab}=\Lambda g_{ab}$ with Λ positive, demand $\operatorname{Ker} L^*=\{0\}$ and hence $\gamma(\,,)$ is positive definite in such spaces. Thus for example, a consistent theory of spin-3/2 particles exists in de Sitter space. Note that we do not have any results for the $\Lambda<0$ case. However, if one considers asymptotically simple space-times in the sense of Penrose, such space-times ($\Lambda<0$) are asymptotically anti-de Sitter (Penrose, 1965a) which have the interesting property that there are no Cauchy surfaces in the space. Standard quantization of fields, such as the one envisaged here, is meaningless in this case.

When $\Lambda=0$, Lemma 3 gives a concrete result, but only for the instances when the space-time admits a $\pi=$const Cauchy surface. For example, Kerr and Schwarzschild space-times admit such a Cauchy slice; so by Lemma 3, $\operatorname{Ker} L^*=\{0\}$ implying $\gamma(\,,)$ is positive definite in these cases. It is remarkable that for $\operatorname{Ker} L^*\neq 0$ the space-time must be flat. However, these flat space-times cannot include Minkowski space-time since there $\lambda_A\in\operatorname{Ker} L^*$ are constant spinors and would not be square integrable unless $\lambda_A=0$. Rather the relevant flat space-times are those which are obtained by suitable identification of Minkowski space so that the resulting space-time is spatially compact. Then the constant spinors $\lambda_A\in\operatorname{Ker} L^*$ would indeed be square integrable. Note that one cannot allow arbitrary identifications of Minkowski space because the spinors at the points being identified must be identified as well. This severely limits the number of permissible identifications of Minkowski space that can be made.[14] Without any assumption about the existence of $\pi=$const slices, we are left to determine, by part (ii) of Lemma 2, the space of solutions of

$$D_{AB}\lambda_C+\left(\pi/\sqrt{2}\right)ABCD\lambda^D=0 \tag{31}$$

Under what situations are there solutions? It turns out (Sen, 1980) that space-times (with "negligible boundary") that admit nonzero solutions of (31) must be algebraically special of Petrov type III [hence including the special cases type N or 0 (flat)]. The type-III or type-N vacuum solutions represent gravitational waves and it is not clear that such space-times can be asymptotically (spatially) flat or that suitable identifications of the space-time can be made to obtain a spatially closed solution. The simplest type-N solution, the plane wave (see, for example, Pirani, 1964), appears to be neither asymptotically flat nor closable (unless flat). Furthermore, Penrose (1965b) has shown that the plane wave solutions do not admit any Cauchy

[14] Explicit examples can be obtained as cross products of flat, compact, Riemannian 3-geometries (see Nowacki, 1934) with time. The 3-geometries that admit constant spinors are the six-parameter family of metrically distinct manifolds with topology $S^1\times S^1\times S^1$.

surface and are therefore uninteresting for our purposes.[15] One may expect this situation to persist even for the general type-N or type-III space-times.

The entire analysis given above focuses on the existence of solutions of $(\tilde{L}^*\lambda)_A = 0$. Do these solutions have any physical significance? We observe that neutrino equation $\nabla^{C'C}\lambda_{C'} = 0$ written in "3+1" form relative to a Cauchy slice Σ is [see (A.33)]

$$(\tilde{L}^*\lambda^\dagger)_A = -(t\cdot\nabla/\sqrt{2})\lambda^\dagger_A$$

The kernel of $\tilde{L}^*$ therefore corresponds to those solutions of the neutrino equation for which $t\cdot\nabla\lambda^\dagger_A = 0$. In a sense these solutions are "zero modes" of the neutrino equation relative to the particular Cauchy slice Σ. Of course, they need not be "zero modes" relative to some other slice. Strictly speaking the notion of "zero modes" is well defined only when we have a timelike Killing field on the space-time. In static space-times, $t\cdot\nabla$ is indeed the "true" time derivative of the field. In that case $\operatorname{Ker} L^*$ gives the "true" zero modes of the neutrino equation. Note that we can conclude that there are *no* normalizable "zero modes" of the neutrino equation in Schwarzschild space-time (since it is asymptotically flat, globally hyperbolic, and admits a $\pi = 0$ slice). This fact can be indeed verified by direct calculation (using, for example, the methods of Chandrasekhar, 1976).

In summary, restricting attention to spatially closed or asymptotically flat space-times (the latter of which we can plausibly assume to have negligible boundary) we have two distinct results. First, $\gamma(,)$ is positive definite in all space-times satisfying $R_{ab} = \Lambda g_{ab}$ with $\Lambda > 0$. Second, $\gamma(,)$ is positive definite in all vacuum globally hyperbolic space-times except for those of Petrov type III, N, or 0 (flat). In particular, if the vacuum space-time admits a $\pi = \text{const}$ Cauchy slice, then $\gamma(,)$ is indefinite only if the space-time is flat.

5. CONCLUSION

Within the restriction to globally hyperbolic Einstein space-times, we have formulated a quantum theory of the free massless spin-3/2 field by specifying the algebra of quantum operators. Although the consideration of a free field is somewhat simplistic, the foregoing analysis of the spin-3/2 algebra brings to light some interesting features.

Unlike the algebra of the quantum electromagnetic field, the spin-3/2 algebra $\mathfrak{A}_0$ has no center. (By a center, in the case of fermions, we refer to a "graded" center generated by elements that anticommute with every gener-

[15] Of course, smaller regions of the plane wave solutions do have a Cauchy surface Σ, but then Σ is incomplete.

ator of $\mathfrak{A}_0$.) Sorkin's study of the electromagnetic field (Sorkin, 1979) shows that the topology of the space-time dictates whether or not the algebra has a nontrivial center. In the spin-3/2 case, however, one cannot hope to find a similar role of the space-time because of the absence of a center. (The Dirac or the neutrino field algebra also has no center.) In fact, it seems unlikely that central elements of an algebra of a Fermi field would reflect the influence of space-time topology. Presumably the central anticommuting "charges" would then have to be expressed as a flux integral of the field over some closed surface in the space-time. But, this is not possible because of the index structure of a Fermi field.

An alternative way to look for analogs of electromagnetic charge operators would be the following. Recall that since $\mathfrak{A}_0$ is a Clifford algebra it can be regarded as a direct sum of two vector spaces $\mathfrak{A}_0^+$ and $\mathfrak{A}_0^-$ called, respectively, the even and odd parts of $\mathfrak{A}_0$. $\mathfrak{A}_0^+(\mathfrak{A}_0^-)$ contains elements which are sums of products of even (odd) number of generators of $\mathfrak{A}_0$. $\mathfrak{A}_0^+$ is in fact a subalgebra of $\mathfrak{A}_0$ (but $\mathfrak{A}_0^-$ is not). Now the physical observables of the theory belong to $\mathfrak{A}_0^+$ and so one might entertain the idea that elements of $\mathfrak{A}_0^+$ which *commute* with every element of $\mathfrak{A}_0^+$, if they exist, are the proper counterparts of the electromagnetic charge operators. However, it turns out that the existence of such elements is tied to the structure of $\mathfrak{A}_0$ as a Clifford algebra in a way which can have little to do with the underlying space-time. It is known, for example, that when $\mathfrak{A}_0$ is a Clifford algebra over a *finite* even-dimensional vector space W (over $\mathbb{C}$), then the center of $\mathfrak{A}_0^+$ is two dimensional spanned by the identity and an element which anticommutes with every element of $\mathfrak{A}_0^-$ (Chevalley, 1954). In our case, however, $\mathfrak{A}_0$ is infinite dimensional and the center of $\mathfrak{A}_0^+$ is trivial.

A quite independent issue that arises in the case of Fermi fields (in contrast with Bose cases) at the algebraic level is the positivity of the anticommutation relations. In the spin-3/2 theory, the apparent indefiniteness of $\gamma(\,,)$ raises the possibility of an indefinite anticommutator. The space-time, in admitting neutrino "zero modes" or not, dictates our ability to "gauge away" the term contributing to the indefiniteness of $\gamma(\,,)$. Thus it is here that the structure of space-time is of relevance. It is remarkable that among spatially closed or asymptotically flat space-times, we find only a small class that admit neutrino "zero modes." Thus $\gamma(\,,)$ is positive in almost all Einstein space-times that may be of physical interest (e.g., vacuum asymptotically flat space-times). One must be cautioned that we have by no means exhausted the list of space-times that admit neutrino "zero modes." By making specific assumptions—that boundary terms are absent, that the space-time satisfies $R_{ab} = \Lambda g_{ab}$—the scope of our method is quite restricted. Further, our method fails to suggest if Einstein spaces with $\Lambda < 0$ admit neutrino "zero modes." A more general method of analysis which would incorporate the role of boundary terms seems desirable.

There are two attitudes one could adopt towards an indefinite $\gamma(\,,\,)$ in the theory. The positivity of the anticommutator is often imposed as a *physical* requirement because one can then obtain a meaningful particle number representation of the algebra and a probabilistic interpretation of transition amplitudes between two states. If we adhere to this requirement, then one must restrict attention to only those space-times which admit a positive $\gamma(\,,\,)$.

An alternative stand would be to consider all possible space-times. Then we admit an indefinite anticommutator and must suitably interpret its consequences. In the spin-3/2 theory, the operators that have a negative anticommutator are those associated with the subspace W_2 of the space of data (see the end of Section 2.5). In the flat space examples, W_2 is two dimensional and its elements are in fact static or "zero-frequency" classical solutions of the spin-3/2 field equation. Therefore, the operators associated with these "zero modes" cannot be decomposed into creation and annihilation parts. In fact no Fock representation of the algebra exists. One would have to consider a new representation of the algebra on an indefinite metric Hilbert space. What significance can one assign to these operators? We have not attempted to answer this question here. Further analysis, particularly of the representation of the algebra, is required. Moreover, to gain some insight, it will be useful to have examples of nonflat vacuum space times admitting neutrino "zero modes."

An interesting feature of the theory that we have not dealt with and that could depend on the structure of the space-time is the existence of spin-3/2 "zero mode" solutions. Their role in the quantum theory is *not* tied to the issue of positivity of $\gamma(\,,\,)$. Indeed, even if $\gamma(\,,\,)$ were positive, such "zero mode" solutions would still contribute to the quantum theory in the manner described by Jackiw and Rebbi (1976) in the context of Dirac field in a background field. In connection with this feature, we note that the space-times for which $\operatorname{Ker} L^* \neq \{0\}$ are precisely those in which one can expect new quantum behavior of the *neutrino* field. Thus our analysis in Section 4 is also germane to this issue.

In summary, the spin-3/2 theory that we have considered provides a simple model of a quantum Fermi field which displays some new behavior in its "interaction" with the underlying space-time. The model and the techniques used to analyze it suggest how one might investigate higher spin Fermi fields with more complicated gauge behavior. Perhaps some features of this theory may be of value to supergravity or supersymmetry theories.

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APPENDIX A: "3+1" DECOMPOSITION OF SPINOR EQUATIONS

Let M be a four-dimensional manifold with a smooth metric g_{ab} of signature $(+---)$. Assume M admits a time function t, a smooth scalar field on M whose gradient is everywhere timelike. Then the one-parameter family of surfaces Σ_t, defined by $t=\text{const}$, are spacelike, three-dimensional surfaces. Let t^a denote the unit normal $(t^a t_a = 1)$ to this family of surfaces and let $\xi^a = N t^a$ be the connecting vector field from each surface to nearby surfaces. That is, $\xi^a \nabla_a t = 1$, whence $t_a = N \nabla_a t$. N is called the lapse function.

In this appendix we shall define spinor fields on a fixed hypersurface Σ_t and show how to obtain an "initial" value formulation of spinor field equations. In other words, the data for the spinor field are specified on the hypersurface Σ_t, possibly subject to some constraint equations, and the evolution equations are given. For a parallel exposition, see Sommers (1980).

To fix notation, we summarize the relevant geometric structure on a fixed hypersurface Σ_t which we shall use in the main discussion. For details, see Geroch (1972). A tensor field $T^{a\cdots b}{}_{c\cdots d}$ on M will be called *spatial* (relative to Σ_t) if its contraction with t^a or t_a vanishes, i.e.,

$$t_a T^{a\cdots b}{}_{c\cdots d} = 0, \ldots, t_b T^{a\cdots b}{}_{c\cdots d} = 0$$

$$t^c T^{a\cdots b}{}_{c\cdots d} = 0, \ldots, t^d T^{a\cdots b}{}_{c\cdots d} = 0$$

There are two spatial tensors of particular interest. The first is the tensor field on Σ_t

$$h_{ab} := g_{ab} - t_a t_b \tag{A.1}$$

which is the induced metric [signature $(---)$] on Σ_t. The contravariant metric is $h^{ab} = g^{ab} - t^a t^b$. (We raise and lower indices with g_{ab}; for spatial tensors, however, raising and lowering with h_{ab} also gives the same result.) The tensor field $h_a{}^b = h_{am} g^{mb} = \delta_a{}^b - t_a t^b$ may be viewed as a projection tensor $(h_a{}^b h_b{}^c = h_a{}^c)$ which projects out the *spatial* part of any vector or covector field on M. Thus, for example

$$\pi_{ab} = h_a{}^m h_b{}^n \nabla_m t_n \tag{A.2}$$

is a spatial tensor field on Σ_t. Here ∇_a is the covariant derivative operator defined by the metric g_{ab}. π_{ab} is the second tensor field of interest; it is the extrinsic curvature of the hypersurface Σ_t. Using $t_a = N\nabla_a t$, it is easy to show that $\pi_{ab} = \pi_{ba}$, i.e., symmetric.

On Σ_t, there is a unique torsion-free derivative operator, denoted by D_a, defined by the spatial metric h_{ab}. The action of D_a on any spatial tensor field $T^{c\cdots d}{}_{e\cdots f}$ may be obtained from the action of ∇_a by the following prescription

$$D_a T^{c\cdots d}{}_{e\cdots f} := h_a{}^b h_m{}^c \cdots h_n{}^d h_e{}^p \cdots h_f{}^q \nabla_b T^{m\cdots n}{}_{p\cdots q} \tag{A.3}$$

In particular, $D_a h_{bc} = 0$

A.1. Spinor Fields on Σ_t. Fix a hypersurface Σ_t. Assign to each point p on Σ_t a complex two-dimensional vector space V equipped with a nondegenerate symplectic form (skew 2-form). Elements of V will be denoted by ξ^A and the symplectic form by ε_{AB}. The complex conjugate vector space associated with V will be denoted by $\bar{V}$, its elements by $\bar{\xi}^{A'}$ and the symplectic form by $\varepsilon_{A'B'}$. ($\bar{V}$ is defined by identifying with V as a set such that $\overline{\alpha\xi^A} = \bar{\alpha}\bar{\xi}^{A'}$, i.e., if $\xi^A \in V$ is identified as $\bar{\xi}^{A'} \in \bar{V}$, then the element $\alpha\xi^A \in V$, $\alpha \in \mathbf{C}$ is to be identified with $\bar{\alpha}\bar{\xi}^{A'} \in \bar{V}$.) The group that preserves the structure on V is $SL(2,\mathbf{C})$ and ξ^A will be called a $SL(2,\mathbf{C})$ spinor at $p \in \Sigma_t$; letting p vary over Σ_t we obtain a spinor field on Σ_t. It is clear that given a $SL(2,\mathbf{C})$ spinor field on M, its restriction to Σ_t is a $SL(2,\mathbf{C})$ spinor field on Σ_t.

To relate tensors to spinors on M, one must fix a metric-preserving isomorphism between the four-dimensional real vector space $W = \{\xi^{AA'} \in V \otimes \bar{V}: \overline{\xi^{AA'}} = \xi^{A'A}$, metric$= \varepsilon_{AB}\varepsilon_{A'B'}\}$ (constructed from the spinor spaces V and $\bar{V}$ at a point p) and the tangent space $T_p(M)$ with metric g_{ab}. We shall assume that such an isomorphism has been fixed.

In order to relate spinor fields on Σ_t to the geometry of Σ_t, we need an additional structure on V which reduces the structure group $SL(2,\mathbf{C})$ to $SU(2)$ (see also Friedman and Sorkin, 1980). This additional structure on V is a distinguished positive definite Hermitian inner product which we denote by $G(,)$:

$$G: V \times V \to \mathbf{C}$$

and

(i) $G(\xi, i\eta) = iG(\xi, \eta)$

(ii) $\overline{G(\xi, \eta)} = G(\eta, \xi)$

(iii) $G(\xi, \xi) \geq 0, \ = 0 \text{ iff } \xi = 0 \qquad \xi, \eta \in V$

where the overbar denotes complex conjugation, (i) and (ii) imply that G is antilinear in its first entry: $G(i\xi,\eta)=-iG(\xi,\eta)$. Consequently, G can be regarded as a *bilinear* map from $\bar{V}\times V$ to $\mathbb{C}$. We introduce the notation

$$G_{A'B}: \bar{V}\times V\to\mathbb{C}$$

$$(\bar{\xi}^{A'},\eta^{A})\mapsto\bar{\xi}^{A'}G_{A'B}\eta^{B} \tag{A.4}$$

In this notation for the Hermitian metric, Hermiticity appears as $G_{A'B}=\bar{G}_{BA'}$. Since the metric is positive definite, its inverse exists, which we denote by $G^{A'B}$ such that

$$G^{AB'}G_{B'C}=\delta_C{}^A \tag{A.5}$$

where $\delta_C{}^A$ is the usual Kronecker delta.

We note that there are three complex vector spaces associated with V: The complex conjugate $\bar{V}$ whose elements are written as $\bar{\xi}^{A'}$, the dual V^* whose elements are written as ξ_A, and the complex conjugate dual $\bar{V}^*$ whose elements are written as $\bar{\xi}_{A'}$. Thus, $G_{B'C}\in\bar{V}^*\otimes V^*$ and $G^{B'C}\in\bar{V}\otimes V$ are tensors over $V,\bar{V},V^*,\bar{V}^*$. If a symplectic structure ε_{AB} is fixed on V, then the ε_{AB} provides an isomorphism between V and V^*:

$$\varepsilon_{AB}: V\to V^*$$

$$\xi^A\mapsto\xi^A\varepsilon_{AB}=\xi_B \tag{A.6}$$

Likewise the inverse ε^{AB}: $V^*\to V$

$$\xi_B\mapsto\varepsilon^{AB}\xi_B=:\xi^A$$

Note that $\varepsilon^{AB}\varepsilon_{CB}=\delta_C{}^A$. Thus we can raise and lower indices with ε_{AB} according to the rule in (A.6). Prime indices are raised and lowered using the corresponding symplectic form $\varepsilon_{A'B'}$ on $\bar{V}$. The introduction of the Hermitian metric $G_{A'B}$ on (V,ε_{AB}) provides an isomorphism between $\bar{V}$ and V^* (and, by Hermiticity, $\bar{V}^*\leftrightarrow V$):

$$G_{A'B}: \bar{V}\to V^*$$

$$\bar{\xi}^{A'}\mapsto\bar{\xi}^{A'}G_{A'B}=:-\xi^{\dagger}_B$$

Note that $(\xi^{\dagger}_B)^{\dagger}=-\xi_B$. To preserve raising and lowering operations with ε_{AB} and $\varepsilon_{A'B'}$, we impose the compatibility condition on $G_{AA'}$

$$\varepsilon^{AB}G_{AA'}G_{BB'}=\varepsilon_{A'B'} \tag{A.7}$$

The structure group of $(V, \varepsilon_{AB}, G_{A'A})$ is $SU(2)$ and elements of (V, ε, G) are $SU(2)$ spinors. The relation between $SU(2)$ spinors and vectors in real three-dimensional Euclidean space emerges as follows. (For details see Friedman and Sorkin, 1980.) A spinor V^{AB} will be called *real* if

$$G_{A'A}G_{B'B}\bar{V}^{A'B'} = V_{AB}$$

Consider the real three-dimensional vector space $E = \{V^{AB}\colon\ V^{AB}$ real and $V^{AB} = V^{(AB)}\}$. There is a natural inner product on E given by

$$h(u, v) := u^{AB}v_{AB}\ \forall u, v \in E$$

The space (E, h) is a real three-dimensional Euclidean space.

We return to the main discussion now. We have considered at each point p of Σ_t the space (V, ε_{AB}). There is a natural Hermitian inner product on (V, ε_{AB}). This is seen as follows. At each point of Σ_t we have the unit normal to vector t^a. In spinor notation, $t^a \equiv t^{A'A}$ is a Hermitian spinor. Since $t_a \equiv t_{A'A}$, the condition that $t^a t_a = 1$ translates into

$$t^{AA'}t_{BA'} = \tfrac{1}{2}\delta_B{}^A \tag{A.8}$$

Set

$$G_{A'B} = \sqrt{2}\, t_{A'B} \tag{A.9}$$

To see that $G_{A'B}$ is indeed positive definite, note that for every $\xi^A \in V$ $\bar{\xi}^{A'}\xi^A \equiv l^a$ is a real null vector at p. (We may choose l^a to be future directed.) Then $\bar{\xi}^{A'}G_{A'B}\xi^B = \sqrt{2}\,\bar{\xi}^{A'}\xi^A t_{A'A} = \sqrt{2}\, l^a t_a \geqslant 0$, $=0$ iff $l^a = 0$. Including this preferred metric on (V, ε_{AB}), we have the result that unprimed spinor fields $T^{A\cdots B}{}_{C\cdots D}$ on M restricted to Σ_t are $SU(2)$ spinor fields on Σ_t.

On M there are also primed spinor fields, for example $\bar{S}_{B'}$. By means of the isomorphism provided by $G_{A'B}$ between primed and unprimed spinors, we may regard primed spinor fields $\bar{S}_{B'}$ on Σ_t as the $SU(2)$ spinor fields $S_B^{\dagger} := \sqrt{2}\, t_B{}^{B'}\bar{S}_{B'}$ on Σ_t. We define spinor fields on Σ_t as the set of all (unprimed) $SU(2)$ spinor fields; denote this set by S_t. Given a spinor field $T^{A'\cdots B'A\cdots B}{}_{C'\cdots D'C\cdots D}$ on M, we obtain a unique spinor field on Σ_t via the rule

$$T^{M\cdots NA\cdots B}{}_{P\cdots QC\cdots D} := \left(-\sqrt{2}\, t^M{}_{A'}\right)\cdots\left(-\sqrt{2}\, t^N{}_{B'}\right)\left(\sqrt{2}\, t^{C'}{}_P\right)\cdots \times\left(\sqrt{2}\, t^{D'}{}_Q\right) T^{A'\cdots B'A\cdots B}{}_{C'\cdots D'C\cdots D}\big|_{\Sigma_t} \tag{A.10}$$

In other words, we convert the primed indices to unprimed ones and restrict the resulting spinor field to Σ_t. Indices on spinor fields on Σ_t will be raised and lowered with ε_{AB}. We shall use the following notation for the $G_{A'B}$ inner product of $SU(2)$ spinors:

$$\xi^{B\dagger}\eta_B := G^{A'B}\bar{\xi}_{A'}\eta_B \tag{A.11}$$

In general for $\phi_{A\cdots B}\in S_t$

$$\phi^{A\cdots B\dagger}\phi_{A\cdots B}=\left(\sqrt{2}\,t^{AA'}\right)\cdots\left(\sqrt{2}\,t^{BB'}\right)\bar{\phi}_{A'\cdots B'}\phi_{A\cdots B} \tag{A.12}$$

which is positive definite.

To express spatial tensors on Σ_t in terms of $SU(2)$ spinors we note that for $S_a\equiv S_{AA'}$ and $t^aS_a=0$ the corresponding unprimed spinor $S_{AB}:=\sqrt{2}\,t_A{}^{A'}S_{A'B}$ is symmetric, i.e., $S_{AB}=S_{(AB)}$. (The skew part of S_{AB} is $\varepsilon_{AB}t^aS_a=0$.) Thus, the rule is to replace each index of a spatial tensor by a pair of symmetrized spinor indices. If $T_{a\cdots b}{}^{c\cdots d}$ is a spatial tensor of type (r,s), then

$$T_{a\cdots b}{}^{c\cdots d}\equiv T_{(AM)\cdots(BN)}{}^{(CP)\cdots(DQ)}$$

Note that since $(\eta_A^\dagger)^\dagger=-\eta_A$, $T^\dagger{}_{\cdots}{}^{\cdots}=(-)^{r+s}T_{\cdots}{}^{\cdots}$.

We end this section by writing the metric h_{ab} and the extrinsic curvature π_{ab} on Σ_t in terms of spinor fields on Σ_t. The spatial metric $h_{ab}=g_{ab}-t_at_b$ is written as $h_{A'AB'B}=\varepsilon_{AB}\varepsilon_{A'B'}-t_{A'A}t_{B'B}$. Define

$$h_{ACBD}:=2t_C{}^{A'}t_D{}^{B'}h_{AA'BB'} \tag{A.13}$$

Using (A.8)

$$h_{ACBD}=-\varepsilon_{B(A}\varepsilon_{C)D} \tag{A.14}$$

Since $t^ah_{ab}=0$, it is easy to show that [also implied by (A.14)]

$$h_{ACBD}=h_{(AC)(BD)} \tag{A.15}$$

and since $h_{ab}=h_{ba}$

$$h_{ACBD}=h_{BDAC} \tag{A.16}$$

The extrinsic curvature $\pi_{ab}\equiv\pi_{A'AB'B}$ corresponds to

$$\pi_{ACBD}:=2t_C{}^{A'}t_D{}^{B'}\pi_{A'AB'B} \tag{A.17}$$

and, like h_{ab}, has the property

$$\pi_{ACBD}=\pi_{(AC)(BD)}=\pi_{(BD)(AC)} \tag{A.18}$$

From (A.18) $\varepsilon^{AB}\pi_{ACBD}=\pi_{ACD}{}^{A}=-\pi_{ADC}{}^{A}$, which implies

$$\pi_{MAB}{}^{M}=\tfrac{1}{2}\varepsilon_{AB}\pi \tag{A.19}$$

where $\pi=\pi_{MN}{}^{MN}=\pi_{MM'}{}^{MM'}=h^{ab}\pi_{ab}=$ trace of extrinsic curvature.

A.2. Derivative Operator. In this section we introduce a "spatial" derivative operator on spinor fields in S_t which refers only to the intrinsic geometry of Σ_t. In particular, its action on tensors on Σ_t (regarded as spinor fields) reduces to (A.3).

Let $\lambda_A \in S_t$ and consider a derivative operator $D_{(AB)}$: $S_t \to S_t$ whose action on λ_A is defined by

$$D_{AB}\lambda_C := \sqrt{2}\, t_{(A}{}^{A'}\nabla_{B)A'}\lambda_C + \left(\pi_{ABCD}/\sqrt{2}\right)\lambda^D \tag{A.20}$$

where $\nabla_{A'A}$ is the spinor form of the torsion-free covariant derivative operator ∇_a on M defined by g_{ab}. The action of D_{AB} on scalar fields ϕ on Σ_t is

$$D_{AB}\phi=\sqrt{2}\, t_{(A}{}^{A'}\nabla_{B)A'}\phi \tag{A.21}$$

and its action on spinors of higher valence is extended by the Leibniz's rule. Thus, for example,

$$\begin{aligned} D_{AB}\varepsilon_{CD} &= \sqrt{2}\, t_{(A}^{A'}\nabla_{B)A'}\varepsilon_{CD} - \frac{\pi_{ABC}{}^{M}}{\sqrt{2}}\varepsilon_{MD} - \frac{\pi_{ABD}{}^{M}}{\sqrt{2}}\varepsilon_{CM} \\ &= 0 - \frac{1}{\sqrt{2}}\left(\pi_{ABCD}-\pi_{ABDC}\right)=0 \end{aligned} \tag{A.22}$$

Thus

$$D_{AB}h_{CDEF}=0 \tag{A.23}$$

It is not hard to show that for any scalar field ϕ on Σ_t

$$[D_{AB}D_{CD}-D_{CD}D_{AB}]\phi=0 \tag{A.24}$$

i.e., D_{AB} is torsion free.

Next we illustrate the action of D_{AB} on spatial tensors. Consider a spatial covector S_a; the spinor field on Σ_t corresponding to S_a is $S_{AB} = \sqrt{2}\, t_{(A}{}^{A'}S_{B)A'}$ ($S_a \equiv S_{AA'}$). We claim

$$D_{CD}S_{AB} = 2t_C{}^{C'}t_B{}^{B'}D_{C'D}S_{B'B} \tag{A.25}$$

where $D_{A'A}S_{B'B} \equiv D_aS_b = h_a{}^m h_b{}^n \nabla_m S_n$. [See equation (A.3).] To show this write $D_{A'A}S_{B'B} = h_{A'A}{}^{M'M}h_{B'B}{}^{N'N}\nabla_{M'M}S_{N'N}$. Then

$$\begin{aligned}
&2t_C{}^{A'}t_D{}^{B'}D_{A'A}S_{B'B}\\
&\quad = 2h_{CA}{}^{PM}h_{BD}{}^{QN}\left(t_P{}^{P'}t_Q{}^{Q'}\nabla_{P'M}S_{Q'N}\right)\\
&\quad = 2h_{CA}{}^{PM}h_{BD}{}^{QN}\left[t_P{}^{P'}\nabla_{P'M}\left(t_Q{}^{Q'}S_{Q'N}\right) - \left(t_p{}^{P'}\nabla_{P'M}T_Q{}^{Q'}\right)S_{Q'N}\right]\\
&\quad = 2h_{CA}{}^{PM}h_{BD}{}^{QN}\left[\left(t_P{}^{P'}/\sqrt{2}\right)\nabla_{P'M}S_{QN} - \pi_{PMQD}t^D{}_{D'}S^{D'}{}_N\right]\\
&\quad = h_{CA}{}^{PM}h_{BD}{}^{QN}\left[\sqrt{2}\,t_{(P}{}^{P'}\nabla_{M)P'}S_{QN} + 2\left(\pi_{PMD(Q}/\sqrt{2}\right)S^D{}_{N)}\right]\\
&\quad = D_{CA}S_{BD}
\end{aligned}$$

In the third equality we have used a convenient expression

$$\pi_{ABCD} = 2\left[t_{(A}{}^{P'}\nabla_{B)P'}t_{Q'(C}\right]t_{D)}{}^{Q'} \tag{A.26}$$

In the fourth equality, the symmetrizations over PM and QN come from the symmetry of the indices in h_{ab}. The last equality uses the definition of $D_{CA}S_{BD}$.

To summarize, the derivative operator D_{AB} defined by (A.20) satisfies the Leibniz rule, is linear, commutes with contraction, kills the spatial metric, is torsion free, and gives the "proper" action on spatial tensors. It is therefore the unique derivative operator on Σ_t defined by h_{ab}.

The Riemann tensor on Σ_t defined by the spatial derivative operator D_a is defined by $D_{[a}D_{b]}S_c = \frac{1}{2}\mathfrak{R}_{abc}{}^dS_d$ for any spatial covector S_a. We would like to have an expression for $[D_{AC}D_{BD} - D_{BD}D_{AC}]\lambda_M$ in terms of the Riemann tensor (in spinor form) for any $\lambda_M \in S_t$. We state the result:

$$[D_{AC}D_{BD} - D_{BD}D_{AC}]\lambda_E = \left[\varepsilon_{AB}D_{M(C}D_{D)}{}^M + \varepsilon_{CD}D_{M(A}D_{B)}{}^M\right]\lambda_E \tag{A.27}$$

$$D_{M(A}D_{B)}{}^M\lambda_C = \tfrac{1}{4}\left[2\mathfrak{R}_{ACBD} - \varepsilon_{AB}\varepsilon_{CD}\tfrac{1}{2}\mathfrak{R}\right]\lambda^D \tag{A.28}$$

where $\mathfrak{R}_{ACBD}:=2t_A{}^{A'}t_B{}^{B'}\mathfrak{R}_{A'CB'D}$ is the spinor form of the Ricci tensor $\mathfrak{R}_{ab}$ and $\mathfrak{R}=\mathfrak{R}_{ab}h^{ab}$ is the scalar curvature. Note that $\mathfrak{R}_{ACBD}=\mathfrak{R}_{(AC)(BD)}=\mathfrak{R}_{BDAC}$ and $\mathfrak{R}_{ACB}{}^{C}=\frac{1}{2}\varepsilon_{AB}\mathfrak{R}$. Contracting the indices B, C in (A.28),

$$D_{M(A}D_{B)}{}^{M}\lambda^{B}=\tfrac{1}{8}\mathfrak{R}\lambda_A \tag{A.29}$$

We end this section with an important property of D_{AB}:

$$(D_{AB}\eta^{\dagger}_{C})=-(D_{AB}\eta_C)^{\dagger}$$

where

$$\begin{aligned}\eta^{\dagger}_{C}&=\sqrt{2}\,t_C{}^{C'}\bar{\eta}_{C'}\\ (D_{AB}\eta_C)^{\dagger}&=(2)^{3/2}t_A{}^{A'}t_B{}^{B'}t_C{}^{C'}D_{A'B'}\bar{\eta}_{C'}\end{aligned} \tag{A.30}$$

To establish this result consider the right-hand side:

$$\begin{aligned}(D_{AB}\eta_C)^{+}&=(2)^{3/2}t_A{}^{A'}t_B{}^{B'}t_C{}^{C'}D_{A'B'}\bar{\eta}_{C'}\\ &=(2)^{3/2}t_A{}^{A'}t_B{}^{B'}t_C{}^{C'}\left[\sqrt{2}\,t_{(A'}{}^{M}\nabla_{B')M}\bar{\eta}_{C'}+\left(\bar{\pi}_{A'B'C'D'}/\sqrt{2}\right)\bar{\eta}^{D'}\right]\\ &=4t_{(A}{}^{A'}t_{B)}{}^{B'}t_C{}^{C'}t_{A'}{}^{M}\nabla_{B'M}\bar{\eta}_{C'}+\pi_{ABCD}t^{DD'}\bar{\eta}_{D'}\\ &=-2t_C{}^{C'}t_{(A}{}^{A'}\nabla_{B)B'}\bar{\eta}_{C'}+\pi_{ABCD}t^{DD'}\bar{\eta}_{D'}\end{aligned} \tag{A.31}$$

In the second step we used the definition of $D_{A'B'}\bar{\eta}_{C'}$, the complex conjugate of equation (A.20); in the third step we used the identity $t_A{}^{A'}t_B{}^{B'}t_C{}^{C'}t_D{}^{D'}\pi_{A'B'C'D'}=\frac{1}{4}\pi_{ABCD}$; and in the last step we simplified the first term using $t^{A'A}t_{A'B}=\frac{1}{2}\delta^{A}{}_{B}$. Similarly $D_{AB}\eta^{\dagger}_{C}$ can be written as $D_{AB}\eta^{\dagger}_{C}=2t_C{}^{C'}t_{(A}{}^{A'}\nabla_{B)A'}\bar{\eta}_{C'}-2[t_{(A}{}^{A'}\nabla_{B)A'}t_{CC'}]\bar{\eta}^{C'}+\pi_{ABCD}t^{DD'}\bar{\eta}_{D'}$. Using the relation [which follows from (A.26)] $\pi_{ABCD}t^{DE'}=-t_{(A}{}^{A'}\nabla_{B)A'}t_C{}^{E'}$ in the second term,

$$D_{AB}\eta^{\dagger}_{C}=2t_C{}^{C'}t_{(A}{}^{A'}\nabla_{B)A'}\bar{\eta}_{C'}-\pi_{ABCD}t^{DD'}\bar{\eta}_{D'} \tag{A.32}$$

Comparing (A.31) and (A.32) leads to (A.30).

A.3. "Initial Value Formulation" of Spinor Equations. In order to express a field equation in a "3+1" form, we need to write the covariant derivative operator $\nabla_{A'A}$ in terms of a spatial derivative operator and a suitable time derivative. The first step is to unprime the primed index in

$\nabla_{A'A}$: $\sqrt{2}\, t_B{}^{A'} \nabla_{AA'} = (\sqrt{2}\,\varepsilon_{AB}/2) t\cdot\nabla + \sqrt{2}\, t_{(B}{}^{A'} \nabla_{A)A'}$ where $t\cdot\nabla = t^{MM'}\nabla_{MM'}$. The first term represents the time derivative[5] and the second represents the "spatial" derivative. When Σ_t has a nonzero extrinsic curvature, $\sqrt{2}\, t_{(B}{}^{A'} \nabla_{A)A'}$ is not the spatial derivative, which refers only to the intrinsic geometry of Σ_t. Rather, as we saw in the previous section, D_{AB} defined by (A.20) gives the spatial derivative. Now we consider specific spinor equations to illustrate the "3 + 1" decomposition.

(i) Neutrino Equation: $\nabla_{C'C}\eta^C = 0$. The datum for the neutrino equation is the restriction of η^A on the surface Σ_t. To obtain the evolution equation on Σ_t we consider the spinor field on Σ_t given by

$$\sqrt{2}\, t_A{}^{C'} \nabla_{C'C}\eta^C = 0$$

which is rewritten as

$$\sqrt{2}\, t_{[A}{}^{C'} \nabla_{C]C'}\eta^C + \sqrt{2}\, t_{(A}{}^{C'} \nabla_{C)C'}\eta^C = 0$$

or

$$\sqrt{2}\,\frac{\varepsilon_{CA}}{2} t\cdot\nabla\,\eta^C + D_{CA}\eta^C + \frac{\pi}{2\sqrt{2}}\eta_A = 0$$

using (A.20) for the second term. Thus, the neutrino equation, expressed in terms of the data on Σ_t is

$$t\cdot\nabla\,\eta_A = -\sqrt{2}\left[D_{AC}\eta^C + \left(\pi/2\sqrt{2}\right)\eta_A\right] \tag{A.33}$$

Note that there are no constraints on the data.

(ii) Maxwell's Equation: $\nabla_{C'C}\phi^{CB} = 0$. The data are given by the spinor field $\phi^{(AB)}$ on Σ_t. The evolution and constraint equations are obtained as follows:

$$\begin{aligned} T_A{}^B &:= \sqrt{2}\, t_A{}^{C'} \nabla_{C'C}\phi^{CB} \\ &= \sqrt{2}\, t_{[A}{}^{C'} \nabla_{C]C'}\phi^{CB} + \sqrt{2}\, t_{(A}{}^{C'} \nabla_{C)C'}\phi^{CB} \\ &= \sqrt{2}\, t\cdot\nabla\,\phi_A{}^B + D_{AC}\phi^{CB} - \frac{\pi_{ACM}{}^B}{\sqrt{2}}\phi^{MC} + \frac{\pi}{2\sqrt{2}}\phi_A{}^B = 0 \end{aligned}$$

Lowering the index B and writing $T_{AB} = T_{(AB)} + \frac{1}{2}\varepsilon_{AB}T_M{}^M$ we obtain $T_{(AB)} = 0$

and $T_M{}^M=0$, which correspond, respectively, to the equations

$$\sqrt{2}\,t\cdot\nabla\,\phi_{AB}+D_{C(A}\phi_{B)}{}^C-\frac{\pi_{C(AB)M}}{\sqrt{2}}\phi^{CM}+\frac{\pi}{2\sqrt{2}}\phi_{AB}=0 \qquad \text{(A.34)}$$

$$D_{AB}\phi^{AB}=0 \qquad \text{(A.35)}$$

(A.34) is the evolution equation and (A.35) is the constraint equation on the data $\phi_{(AB)}$ and Σ_t.

(iii) Spin-3/2 Equation: $\nabla_{A'}{}^A\psi_{B'CA}=0$. The data for the field $\psi_{B'CA}$ are given by the spinor field $\psi_{A(BC)}:=\sqrt{2}\,t_A{}^{A'}\psi_{A'(BC)}$ on Σ_t. The constraint and the evolution equations are given by

$$T_{ABC}:=2t_A{}^{A'}t_B{}^{B'}\nabla_{A'}{}^M\psi_{B'CM}=0$$

In terms of the data,

$$T_{ABC}=-\frac{t\cdot\nabla}{\sqrt{2}}\psi_{BCA}-D_{AM}\psi_{BC}{}^M+\frac{\pi_{AMB}{}^D}{\sqrt{2}}\psi_{DC}{}^M+\frac{\pi_{AMCD}}{\sqrt{2}}\psi_B{}^{DM}$$
$$-\frac{\pi}{2\sqrt{2}}\psi_{BCA}+\psi^M{}_{CA}\frac{D_{MB}N}{N} \qquad \text{(A.36)}$$

Now T_{ABC} can be decomposed into irreducible pieces according to

$$T_{ABC}=T_{(ABC)}+\tfrac{2}{3}\varepsilon_{C(A}\eta_{B)}+\tfrac{2}{3}\lambda_{(A}\varepsilon_{C)B}$$

where $\eta_B:=\varepsilon^{CA}T_{ABC}$, $\eta_C:=\varepsilon^{AB}T_{ABC}$. Since $T_{ABC}=0$, each irreducible piece must vanish separately. $T_{(ABC)}=0$, $\lambda_A=0$, and $\eta_A=0$ give, respectively, the equations (9.1), (9.2), and (9.3) in the main text.

APPENDIX B: CAUSAL PROPAGATION OF SPIN-3/2 FIELD

In this section we show that the characteristic surfaces of the spin-3/2 equation $\nabla_{A'}{}^A\psi_{B'AC}=0$ are *null* (which then suggests that the propagation of the field is causal).

We follow the method of Madore and Tait (1973) where characteristic surfaces are viewed as surfaces across which there can exist discontinuities in the highest-order derivatives appearing in the field equation. Let C be a smooth hypersurface in an open region U in M. C can be defined by some smooth function ϕ in U as the surface $\phi=0$. Then $\xi_a = \nabla_a\phi$ is normal to C. Now C divides U into two regions U^+ (for $\phi\geqslant 0$) and U^- (for $\phi<0$).

Next, consider a spin-3/2 field $\psi_{A'BC}$ in U, satisfying the field equation $\nabla_{A'}{}^{A}\psi_{B'AC}=0$. We wish to obtain expressions for possible discontinuities across C of $\nabla_{A'}{}^{A}\psi_{B'AC}$. To do this denote by $\psi^{\pm}_{A'BC}$ the field $\psi_{A'BC}$ in the regions $U^{\pm}$. The discontinuity of the field across C will be denoted by

$$[\psi_{A'BC}]:=\psi^{+}_{A'BC}-\psi^{-}_{A'BC} \tag{B.1}$$

By extending $\psi^{\pm}_{A'BC}$ smoothly (but otherwise arbitrarily) into $U^{\mp}$, $[\psi_{A'BC}]$ becomes a smooth spinor field in U. Consequently, in the neighborhood of C, there exists a smooth spinor field $K_{A'BC}$ on U such that

$$[\psi_{A'BC}]=\phi K_{A'BC} \tag{B.2}$$

The discontinuity of the first derivative of the field is

$$[\nabla_{MM'}\psi_{A'BC}]|_C=(\nabla_{MM'}\phi)|_C K_{A'BC}|_C=\xi_{MM'}K_{A'BC}|_C \tag{B.3}$$

The discontinuity occurring in the highest-order derivative in the field equation, viz., $\nabla_{A'}{}^{A}\psi_{B'AC}$ is then

$$\left[\nabla_{M'}{}^{M}\psi_{A'BM}\right]=\xi_{M'}{}^{M}K_{A'BM}|_C=0 \tag{B.4}$$

The second equality follows from the field equation. Equation (B.4) now readily implies ξ_a is null: $\xi_{M'}{}^{M}K_{A'BM}|_C=0\Rightarrow\xi_D{}^{M'}\xi_{M'}{}^{M}K_{A'BM}|_C=0\Rightarrow(\xi^a\xi_a)K_{A'BD}|_C=0\Rightarrow\xi^a\xi_a=0$. Thus the normal to C is null, i.e., C is a null surface generated by ξ^a.

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